Analysis III (BAUG) **Assignment 6**

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Fourier expansions and IBVP

Question 1

Solve the following wave equations. (At least 2 exercises)

(i)

$$\begin{aligned}
\text{PDE}: &\quad u_{tt}(x,t) = 2u_{xx}(x,t) && \text{for } 0 < x < 2\pi \text{ and } t > 0 \\
\text{BC}: &\quad u(0,t) = 0, \;\; u(2\pi,t) = 0 && \text{for } t \geq 0 \\
\text{IC}: &\quad u(x,0) = \sin(\tfrac{x}{2}) + \sin(\tfrac{5x}{2}) && \text{for } 0 \leq x \leq 2\pi \\
&\quad u_t(x,0) = 4\sin(3x) && \text{for } 0 \leq x \leq 2\pi
\end{aligned}$$

Solution:
Here the initial conditions are already a sine expansion with $c_1 = c_5 = 1$, $b_6 = 4$ and $a_i = b_j = 0$ for other values. Also, $c^2 = 2$, so $c = \sqrt{2}$. Using the general formula, we get that the solution of this PDE is

$$u(x,t) = \sin(\tfrac{x}{2})\cos(\tfrac{\sqrt{2}t}{2}) + \sin(\tfrac{5x}{2})\cos(\tfrac{5\sqrt{2}t}{2}) + \tfrac{4}{3\sqrt{2}}\sin(3x)\sin(3\sqrt{2}t)$$

(ii)

$$\begin{aligned}
\text{PDE}: &\quad u_{tt}(x,t) = 4u_{xx}(x,t) && \text{for } 0 < x < 10 \text{ and } t > 0 \\
\text{BC}: &\quad u(0,t) = 0, \;\; u(10,t) = 0 && \text{for } t \geq 0 \\
\text{IC}: &\quad u(x,0) = 2\sin(3\pi x) - 4\cos(\tfrac{(2x+5)\pi}{10}) && \text{for } 0 \leq x \leq 10 \\
&\quad u_t(x,0) = 4\cos^2(\pi x - \tfrac{\pi}{4}) - 2 && \text{for } 0 \leq x \leq 10
\end{aligned}$$

Solution:
Using $\cos(t + \frac{\pi}{2}) = -\sin(t)$, we see that $u(x,0) = 2\sin(3\pi x) + 4\sin(\frac{2x\pi}{10})$, thus $c_{30} = 2$ and $c_2 = 4$.

Also, $\cos(2t) = 2\cos^2(t) - 1$ and $\sin(t) = \cos(t - \pi/2)$ imply that $u_t(x,0) = 2\cos(2\pi x - \frac{\pi}{2}) = 2\sin(2\pi x)$, meaning $b_{20} = 2$.

Also, $c^2 = 4$, so $c = 2$ here. Now the formula gives the following solution of the PDE:

$$u(x,t) = 4\sin(\tfrac{2\pi x}{10})\cos(\tfrac{2\pi\cdot 2t}{10}) + 2\sin(3\pi x)\cos(3\pi\cdot 2t) + \tfrac{2}{2\pi\cdot 2}\sin(2\pi x)\sin(2\pi\cdot 2t)$$

(iii)

$$\begin{aligned}
\text{PDE}: &\quad u_{tt}(x,t) = 3u_{xx}(x,t) && \text{for } 0 < x < 6 \text{ and } t > 0\\
\text{BC}: &\quad u(0,t) = 0, \;\; u(6,t) = 0 && \text{for } t \geq 0\\
\text{IC}: &\quad u(x,0) = \sin(2\pi x) && \text{for } 0 \leq x \leq 6\\
&\quad u_t(x,0) = 3 && \text{for } 0 \leq x \leq 6
\end{aligned}$$

Solution:
$u(x,0)$ is already a sine expansion: $c_{12} = 1$, $c_n = 0$ otherwise.

The Fourier coefficients of $u_t(x,0)$ are the following:

$$b_n = \frac{2}{6}\int_0^6 3\sin(\tfrac{n\pi x}{6})\,dx = \tfrac{2}{6}\left[\tfrac{-3\cdot 6}{n\pi}\cos(\tfrac{n\pi x}{6})\right]_0^6 = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{12}{\pi n} & \text{if } n \text{ is odd.} \end{cases}$$

$c^2 = 3$, so $c = \sqrt{3}$, and hence the formula gives the following solution:

$$u(x,t) = \sin(2\pi x)\cos(2\pi\cdot\sqrt{3}t) + \textstyle\sum_{k=0}^{\infty} \frac{12}{\pi(2k+1)}\cdot\frac{6}{(2k+1)\pi\cdot\sqrt{3}}\sin(\frac{(2k+1)\pi}{6}x)\sin(\frac{(2k+1)\pi\cdot\sqrt{3}}{6}t)$$

(iv)

$$\begin{aligned}
\text{PDE}: &\quad u_{tt}(x,t) = u_{xx}(x,t) && \text{for } 0 < x < \pi \text{ and } t > 0\\
\text{BC}: &\quad u(0,t) = 0, \;\; u(\pi,t) = 0 && \text{for } t \geq 0\\
\text{IC}: &\quad u(x,0) = x && \text{for } 0 \leq x \leq \pi\\
&\quad u_t(x,0) = x + 3 && \text{for } 0 \leq x \leq \pi
\end{aligned}$$

Solution:
The Fourier sine coefficients of $u(x,0)$ are the following:

$$c_n = \frac{2}{\pi}\int_0^\pi x\sin(nx)\,dx = \left[-\frac{2}{n\pi}x\cos(nx)\right]_0^\pi + \frac{2}{n\pi}\int_0^\pi \cos(nx)\,dx$$
$$= \left[-\frac{2}{n\pi}x\cos(nx)\right]_0^\pi + \left[-\frac{2}{n^2\pi}\sin(nx)\right]_0^\pi = \begin{cases} -\frac{2}{n} & \text{if } n \text{ is even} \\ \frac{2}{n} & \text{if } n \text{ is odd.} \end{cases}$$

The same computation as above shows that the Fourier sine coefficients of the function $\phi(x) = 3$ on $[0,\pi]$ are

$$d_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{12}{\pi n} & \text{if } n \text{ is odd,} \end{cases}$$

so the Fourier sine coefficients of $u_t(x,0)$ are

$$b_n = c_n + d_n = \begin{cases} -\frac{2}{n} & \text{if } n \text{ is even} \\ \frac{12+2\pi}{\pi n} & \text{if } n \text{ is odd,} \end{cases}$$

Furthermore, $c^2 = 1$, so $c = 1$ and the solution of the PDE here (using the values b_n, c_n above) is:

$$u(x,t) = \textstyle\sum_{n=1}^\infty c_n \sin(nx)\cos(nt) + \sum_{n=1}^\infty \frac{b_n}{n}\sin(nx)\sin(nt)$$

Question 2

Use d'Alembert's formula for the following problems about wave equations on infinite strings. (At least 2 exercises)

(i)

$$\begin{aligned} \text{PDE}: \quad & u_{tt}(x,t) = u_{xx}(x,t) && \text{for } -\infty < x < \infty \text{ and } t > 0 \\ \text{IC}: \quad & u(x,0) = x && \text{for } -\infty < x < \infty \\ & u_t(x,0) = \cos(x) && \text{for } -\infty < x < \infty \end{aligned}$$

Compute $u(x,t)$ for all x and all $t \geq 0$.

Solution:
Here $c^2 = 1$ so $c = 1$. For $f(x) = u(x,0)$ and $g(x) = u_t(x,0)$, d'Alembert's formula gives

$$\begin{aligned} u(x,t) &= \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds \\ &= \frac{x-t+x+t}{2} + \frac{1}{2}\int_{x-t}^{x+t} \cos(s)\,ds = x + \frac{1}{2}(\sin(x+t) - \sin(x-t)). \end{aligned}$$

(ii)

$$\begin{aligned} \text{PDE}: \quad & u_{tt}(x,t) = u_{xx}(x,t) && \text{for } -\infty < x < \infty \text{ and } t > 0 \\ \text{IC}: \quad & u(x,0) = \begin{cases} 8x - 2x^2 & \text{for } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases} \\ & u_t(x,0) = \begin{cases} 16 & \text{for } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Compute $u(11,3)$ and $u(5,2)$.

[Exam question, 2013]

Solution:
Again we use d'Alembert with the usual functions f and g for the initial conditions. $c^2 = c = 1$.

$$u(11,3) = \frac{f(14)+f(8)}{2} + \frac{1}{2}\int_8^{14} g(s)\,ds = 0$$

because f and g are 0 on the interval $[8,14]$.

$$u(5,2) = \frac{f(7)+f(3)}{2} + \frac{1}{2}\int_3^7 g(s)\,ds = \frac{0+6}{2} + \frac{1}{2}\int_3^4 16\,ds = 3 + 8 = 11$$

(iii)

$$
\begin{aligned}
\text{PDE}: \quad & u_{tt}(x,t) = u_{xx}(x,t) && \text{for } -\infty < x < \infty \text{ and } t > 0 \\
\text{IC}: \quad & u(x,0) = 0 && \text{for } -\infty < x < \infty \\
& u_t(x,0) = \begin{cases} 1 & \text{for } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Compute $u(x,t)$ for all x and all $t \geq 0$.

[Exam question, 2008]

Solution:
Once again, $c^2 = c = 1$.

$$
u(x,t) = \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds = \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds
$$

$$
= \begin{cases}
0 & \text{if } x-t \le x+t < -1 \\
\frac{1+x+t}{2} & \text{if } x-t < -1 \text{ and } -1 \le x+t \le 1 \\
1 & \text{if } x-t < -1 \text{ and } 1 < x+t \\
t & \text{if } -1 \le x-t \le x+t \le 1 \\
\frac{1-x+t}{2} & \text{if } -1 \le x-t \le 1 \text{ and } 1 < x+t \\
0 & \text{if } 1 < x-t \le x+t.
\end{cases}
$$

(iv)

$$
\begin{aligned}
\text{PDE}: \quad & u_{tt}(x,t) = u_{xx}(x,t) && \text{for } -\infty < x < \infty \text{ and } t > 0 \\
\text{IC}: \quad & u(x,0) = \frac{1}{x^2+1} && \text{for } -\infty < x < \infty \\
& u_t(x,0) = \frac{1}{x^2}
\end{aligned}
$$

Compute $u(x,t)$ for all x and all $t \geq 0$.

Solution:
Once again, $c^2 = c = 1$. We have $\int_{x-t}^{x+t} \frac{1}{x^2}dx = \frac{1}{x-t} - \frac{1}{x+t} = \frac{-2t}{x^2-t^2}$.

Thus

$$u(x,t) = \frac{1}{2}\left(\frac{1}{x^2+2tx+t^2+1} - \frac{1}{x^2-2tx+t^2+1}\right) + \frac{-t}{x^2-t^2}.$$

Question 3

Let $f(x) = -2x$ for $x \in [0,10]$. Choose the correct Fourier series (sines, cosines or normal one) and write f as a trigonometric series.

Solution:
Since f is defined on an interval of the form $[0,L]$ and not $[-L,L]$ we will not use the normal Fourier series. Since $f(0)=0$, but $f'(0)=2\neq 0$, we can only use the Fourier sine series.

The coefficients are as follows:

$$b_n = \frac{2}{10}\int_0^{10} -2x\sin(\frac{n\pi x}{10})\,dx = \left[\frac{4}{n\pi}x\cos(\frac{n\pi x}{10})\right]_0^{10} - \frac{4}{n\pi}\int_0^{10}\cos(\frac{n\pi x}{10})\,dx$$
$$= \left[\frac{4}{n\pi}x\cos(\frac{n\pi x}{10})\right]_0^{10} + \left[\frac{40}{n^2\pi^2}\sin(\frac{n\pi x}{10})\right]_0^{10} = \begin{cases}\frac{40}{n\pi} & \text{if } n \text{ is even}\\ -\frac{40}{n\pi} & \text{if } n \text{ is odd.}\end{cases}$$

Question 4

Solve the following IBVP.

$$\begin{aligned}
\text{PDE}: &\quad u_t(x,t) = u_{xx}(x,t) &&\text{for } 0<x<10 \text{ and } t>0\\
\text{BC}: &\quad u(0,t)=15,\ \ u(10,t)=35 &&\text{for } t\geq 0\\
\text{IC}: &\quad u(x,0)=15 &&\text{for } 0\leq x<10
\end{aligned}$$

Solution:
We will use the superposition principle. Firstly, we will find a solution that does not depend on the time that satisfy the PDE and the boundary condition. A function f of the variable x that satisfy $f_{xx}=0$ (because f does not depend on t) has the form $f(x)=Ax+B$. By $f(0)=15$ and $f(10)=35$ we get $f(x)=15+2x$.

It is important to remember that f is the steady-state solution of the above IBVP. That means that regardless of the initial condition IC, any solution of the above IBPV will look like f after an infinite amount of time.

Suppose that v satisfy the following IBVP:

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = u_{xx}(x,t) && \text{for } 0 < x < 10 \text{ and } t > 0 \\ \text{BC}: &\quad u(0,t) = u(10,t) = 0 && \text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = 15 - f(x) && \text{for } 0 \leq x < 10 \end{aligned}$$

Then $u = v + f$ is a solution of the original IBVP. But we know how to find such a v. Moreover, we have that $v(x,0) = 15 - f(x) = 15 - 15 - 2x = -2x$, and the sine Fourier series for $-2x$ was computed in exercise 5.

Therefore

$$v(x,t) = \sum_{n=1}^{\infty} (-1)^n \frac{40}{n\pi} e^{-(\frac{n\pi}{10})^2 t} \sin(\frac{n\pi}{10} x).$$

As a result, we get a solution of the original IBVP, namely:

$$u(x,t) = v(x,t) + f(x) = 2x + 15 + \sum_{n=1}^{\infty} (-1)^n \frac{40}{n\pi} e^{-(\frac{n\pi}{10})^2 t} \sin(\frac{n\pi}{10} x).$$

Chain rule training

The goal of this part of exercises it to understand how to the change of variables can allow us to solve new IVPs, that is to solve Question 8 (which is the hard question of this exercise sheet). If you solve Question 8, you don't need to do any other question. However, Questions 5,6 and 7 are a training to Question 8. You should do as many point of them as you need to feel confident with the topics presented.

Question 5

Let $u(x,y)$ be a function satisfying $u_{xx} = u_{tt}$. For each of the following change of coordinates, choose the correct PDE satisfied by u with respect to the new coordinates.

(i) $\xi = 4x, \eta = 5t$.

- ☐ $u_{\xi\xi} = u_{\eta\eta}$;
- ☐ $4u_{\xi\xi} = 5u_{\eta\eta}$;
- ☐ $16u_{\xi\xi} - 40u_{\xi\eta} + 25u_{\eta\eta} = 0$;
- ☐ $16u_{\xi\xi} - 25u_{\eta\eta} = 0$.**Correct**

Solution:
Since $\frac{\partial\xi}{\partial x} = 4$, $\frac{\partial\xi}{\partial t} = 0$, $\frac{\partial\eta}{\partial x} = 0$ $\frac{\partial\eta}{\partial t} = 5$, we have $u_{xx} = 16u_{\xi\xi}$ and $u_{tt} = 25u_{\eta\eta}$. Thus the correct answer is $16u_{\xi\xi} - 25u_{\eta\eta} = 0$.

(ii) $\xi = x + 3t, \eta = t$.

- ☐ $u_{\xi\xi} = u_{\eta\eta}$;
- ☐ $-8u_{\xi\xi} = 6u_{\xi\eta} + u_{\eta\eta}$;**Correct**
- ☐ $9u_{\xi\xi} - u_{\eta\eta} = 0$;
- ☐ $u_{\xi\xi} = 9u_{\eta\eta} + 6u_{\eta\xi}$.

Solution:
We have $u_{xx} = u_{\xi\xi}$ and $u_{tt} = 9u_{\xi\xi}+6u_{\xi\eta}+u_{\eta\eta}$. Thus the second answer is correct.

(iii) $\xi = 2x, \eta = xt$

- ☐ $\xi^2 u_{\xi\xi} = \eta^2 u_{\eta\eta}$;
- ☐ $\frac{-8}{3}\xi u_{\xi\xi} = 4\frac{\xi}{\eta}u_{\xi\eta} + u_{\eta\eta}$;
- ☐ $4u_{\xi\xi} + 4\eta^2 u_{\xi\eta} + \left(\left(\frac{1}{2\xi}\right)^2 - \eta^2\right) u_{\eta\eta} = 0$;
- ☐ $4(u_{\xi\xi} + \frac{2\eta}{\xi}u_{\xi\eta}) + \left(\left(\frac{2\eta}{\xi}\right)^2 - \left(\frac{1}{2}\xi\right)^2\right) u_{\eta\eta} = 0$.**correct**

Solution:
Consider the following: $\frac{\partial\xi}{\partial x} = 2$, $\frac{\partial\xi}{\partial t} = 0$, $\frac{\partial\eta}{\partial x} = t$ and $\frac{\partial\eta}{\partial t} = x$. Since $x = \frac{1}{2}\xi$ and

$t = \frac{2\eta}{\xi}$, we have $\frac{\partial\eta}{\partial x} = \frac{2\eta}{\xi}$ and $\frac{\partial\eta}{\partial t} = \frac{1}{2}\xi$. Thus:

$$\begin{aligned}
u_x &= 2 \cdot u_\xi + \frac{2\eta}{\xi} \cdot u_\eta = 2u_\xi + \frac{2\eta}{\xi} u_\eta; \\
u_t &= 0 \cdot u_\xi + \frac{1}{2}\xi \cdot u_\eta = \frac{1}{2}\xi u_\eta; \\
u_{xx} &= \frac{\partial}{\partial x}\left(2u_\xi + \frac{2\eta}{\xi}u_\eta\right) = \frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi}\left(2u_\xi + \frac{2\eta}{\xi}u_\eta\right) + \frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta}\left(2u_\xi + \frac{2\eta}{\xi}u_\eta\right) = \\
&= 2\left(2u_{\xi\xi} - 2\frac{\eta}{\xi^2}u_\eta + 2\frac{\eta}{\xi}u_{\eta\xi}\right) + 2\frac{\eta}{\xi}\left(2u_{\xi\eta} + \frac{2}{\xi}u_\eta + 2\frac{\eta}{\xi}u_{\eta\eta}\right) = \\
&= 4\left(u_{\xi\xi} + 2\frac{\eta}{\xi}u_{\xi\eta} + \frac{\eta^2}{\xi^2}u_{\eta\eta}\right) \\
u_{tt} &= \frac{\partial}{\partial t}\left(\frac{1}{2}\xi u_\eta\right) = \frac{\partial\xi}{\partial t}\frac{\partial}{\partial\xi}\left(\frac{1}{2}\xi u_\eta\right) + \frac{\partial\eta}{\partial t}\frac{\partial}{\partial\eta}\left(\frac{1}{2}\xi u_\eta\right) = \\
&= \frac{1}{4}\xi^2 u_{\eta\eta}
\end{aligned}$$

Thus the PDE translates as $u_{\xi\xi} + \frac{\eta}{\xi}u_{\xi\eta} + \left(\frac{\eta^2}{\xi^2} - \frac{1}{16}\xi^2\right)u_{\eta\eta} = 0$.

Question 6

Consider a vibrating infinite string that vibrates with propagation speed c, where c represents the speed of light (for instance the wave caused by a light beam). Suppose that an observer is moving next to it with constant speed v (starting from $x = 0$ at time $t = 0$). If we consider Einstein's theory of relativity, the change of coordinates between the system of the string (coordinates x, t) and the observer (coordinates ξ, η) is $\xi = \gamma(x - vt)$ and $\eta = \gamma\left(t - \frac{vx}{c^2}\right)$, where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is a constant. What is the PDE satisfied by the string from the point of view of the observer?

Solution:
As before, we compute the derivatives of the coordinates. We have $\frac{\partial\xi}{\partial x} = \gamma$, $\frac{\partial\xi}{\partial t} = -\gamma v$,

$\frac{\partial \eta}{\partial x} = -\frac{\gamma v}{c^2}$ and $\frac{\partial \eta}{\partial t} = \gamma$. Thus we get

$$\begin{aligned}
u_x &= \gamma u_\xi - \frac{\gamma v}{c^2} u_\eta \\
u_t &= -\gamma v u_\xi + \gamma u_\eta \\
u_{xx} &= \gamma^2 u_{\xi\xi} - \frac{\gamma^2 v}{c^2} u_{\eta\xi} - \frac{\gamma^2 v}{c^2} u_{\xi\eta} + \left(\frac{\gamma v}{c^2}\right)^2 u_{\eta\eta} = \gamma^2 \left(u_{\xi\xi} - 2\frac{v}{c^2} u_{\xi\eta} + \left(\frac{v}{c^2}\right)^2 u_{\eta\eta}\right) \\
u_{tt} &= \gamma^2 v^2 u_{\xi\xi} - v\gamma^2 u_{\eta\xi} - v\gamma^2 u_{\xi\eta} + \gamma^2 u_{\eta\eta} = \gamma^2 \left(v^2 u_{\xi\xi} - 2v u_{\eta\xi} + u_{\eta\eta}\right)
\end{aligned}$$

Thus from $u_{tt} = c^2 u_{xx}$, we get

$$\begin{aligned}
&\gamma^2 \left(v^2 u_{\xi\xi} - 2v u_{\eta\xi} + u_{\eta\eta}\right) = \gamma^2 c^2 \left(u_{\xi\xi} - 2\frac{v}{c^2} u_{\xi\eta} + \left(\frac{v}{c^2}\right)^2 u_{\eta\eta}\right) \\
\Leftrightarrow\ & v^2 u_{\xi\xi} - 2v u_{\eta\xi} + u_{\eta\eta} = c^2 u_{\xi\xi} - 2v u_{\xi\eta} + \frac{v^2}{c^2} u_{\eta\eta} \\
\Leftrightarrow\ & \left(1 - \frac{v^2}{c^2}\right) u_{\eta\eta} = (c^2 - v^2) u_{\xi\xi} \\
\Leftrightarrow\ & u_{\eta\eta} = c^2 u_{\xi\xi}.
\end{aligned}$$

In particular, we observe exactly the same wave equation. This is because in general relativity, the speed of light is constant in all systems of reference, thus a wave with propagation speed c looks exactly the same in all systems of reference.

Question 7

Consider the following IVP.

PDE :	$u_{tt}(x,t) = u_{xx}$	for $-\infty < x < \infty$ and $t > 0$
IC :	$u(x,0) = f(x)$	for $-\infty < x < \infty$ and $t > 0$
	$u_t(x,0) = g(x)$	for $-\infty < x < \infty$ and $t > 0$

For each coordinate change of exercise 5, write the corresponding IVP for $v(\xi,\eta) = u(\xi(x,t),\eta(x,t))$.

Solution:

(i) $\xi = 4x, \eta = 5t$. We know that the PDE is $16v_{\xi\xi} = 25v_{\eta\eta}$. Moreover, we have that $x = \frac{1}{4}\xi$ and $t = \frac{1}{5}\eta$. Thus $u(x,0) = f(x)$ is equivalent to $v(\frac{1}{4}\xi, 0) = f(\frac{1}{4}\xi)$. It is clear that this is equivalent to $v(\xi,0)=f(\xi)$. For the first derivative, we have that $u_t=5v_\eta$. Thus we obtain:

$$\begin{aligned} \text{PDE}: \quad & 16v_{\xi\xi}=25v_{\eta\eta} && \text{for } -\infty<x<\infty \text{ and } t>0\\ \text{IC}: \quad & v(\xi,0)=f(\xi) && \text{for } -\infty<x<\infty \text{ and } t>0\\ & v_\eta(\xi,0)=\frac{1}{5}g(\xi) && \text{for } -\infty<x<\infty \text{ and } t>0 \end{aligned}$$

(ii) $\xi=x+3t,\eta=t$. We know that the PDE is $-8v_{\xi\xi}=6v_{\xi\eta}+v_{\eta\eta}$. As before, we obtain $x=\xi-3\eta$, and thus, since $\eta=t$, $v(\xi,0)=f(\xi)$. Consider u_t. We know that $u_t=-3v_\xi+v_\eta$, thus $u_t(x,0)=g(x)$ is equivalent to $-3v_\xi(\xi-0,0)+v_\eta(\xi-0,0)=g(\xi-0)$. This, however, can be improved. Indeed, consider the function $v(\xi,0)=f(\xi)$ as a function of the only variable ξ. We have that $v_\xi(\xi,0)=f'(\xi)$. Thus we obtain:

$$\begin{aligned} \text{PDE}: \quad & -8v_{\xi\xi}=6v_{\xi\eta}+v_{\eta\eta} && \text{for } -\infty<x<\infty \text{ and } t>0\\ \text{IC}: \quad & v(\xi,0)=f(\xi) && \text{for } -\infty<x<\infty \text{ and } t>0\\ & v_\eta(\xi,0)=g(\xi)+f'(\xi) && \text{for } -\infty<x<\infty \text{ and } t>0 \end{aligned}$$

(iii) $\xi=2x,\eta=xt$. We know that the PDE is $u_{\xi\xi}+\frac{\eta}{\xi}u_{\xi\eta}+\left(\frac{\eta^2}{\xi^2}-\frac{1}{16}\xi^2\right)u_{\eta\eta}=0$. Moreover we have that $x=\frac{1}{2}\xi$ and $t=\frac{2\eta}{\xi}$. Thus $u(x,0)=f(x)$ translates as $v(\frac{1}{2}\xi,0)=f(\frac{1}{2}\xi)$. Thus also in this case we have $v(\xi,0)=f(\xi)$. Consider $u_t=\frac{1}{2}\xi v_\eta$. We have that $u_t(x,0)=g(x)$ translates as $\frac{1}{4}\xi v_\eta(\frac{1}{2}\xi,0)=g(\frac{1}{2}\xi)$, thus $v_\eta(\xi,0)=\frac{2}{\xi}g(\xi)$. We obtain:

$$\begin{aligned} \text{PDE}: \quad & u_{\xi\xi}+\frac{\eta}{\xi}u_{\xi\eta}+\left(\frac{\eta^2}{\xi^2}-\frac{1}{16}\xi^2\right)u_{\eta\eta}=0 && \text{for } -\infty<x<\infty \text{ and } t>0\\ \text{IC}: \quad & v(\xi,0)=f(\xi) && \text{for } -\infty<x<\infty \text{ and } t>0\\ & v_\eta(\xi,0)=\frac{2}{\xi}g(\xi) && \text{for } -\infty<x<\infty \text{ and } t>0 \end{aligned}$$

Question 8

This question is harder then the other ones. If you have problems with it, Questions 5 – 7 may be of help.

(i) Consider a vibrating infinite string, and suppose that an observer is moving next to it with constant speed v (starting from $x = 0$ at time $t = 0$). What is the PDE satisfied by the string from the point of view of the observer? (Use Newtonian physic)

Solution:

We know that the vibrating string satisfy the PDE $u_{tt} = c^2 u_{xx}$ for some c. Let ξ and η denote the coordinate in the reference system of the observer. That is, $\xi = 0$ represents the place where the observer stand and η represents the time. Since the observer moves at constant speed v, and starts at time $t = 0$ at the position $x = 0$, we have that it's position after an amount of time t_1 is passed is exactly vt_1. Thus at the time $t_1$the coordinate $x = vt_1$ and $\xi = 0$ describe the same point. Thus $\xi(x, t) = x - vt$. Since the time η is the same for all reference systems (at least in Newtonian physics), we have $\eta(x, t) = t$. In order to express the PDE that the string satisfy from the perspective of the observer, we need to express the derivatives u_{xx} and u_{tt} in terms of the new coordinates ξ, η.

Observe that $\frac{\partial \xi}{\partial x} = 1$, $\frac{\partial \xi}{\partial t} = -v$, $\frac{\partial \eta}{\partial x} = 0$, and $\frac{\partial \eta}{\partial t} = 1$. Thus we have

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = u_\xi,$$
$$u_t = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} = -vu_\xi + u_\eta.$$

Second derivatives:

$$\begin{aligned} u_{xx} =& \frac{\partial (u_x)}{\partial x} = \frac{\partial (u_x)}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial (u_x)}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = \frac{\partial u_\xi}{\partial \xi} = u_{\xi\xi}, \\ u_{tt} =& \frac{\partial (u_t)}{\partial \xi} \cdot \frac{\partial \xi}{\partial t} + \frac{\partial (u_t)}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} = -v\frac{\partial(-vu_\xi + u_\eta)}{\partial \xi} + \frac{\partial(-vu_\xi + u_\eta)}{\partial \eta} = \\ &= v^2 u_{\xi\xi} - vu_{\eta\xi} - vu_{\xi\eta} + u_{\eta\eta} = v^2 u_{\xi\xi} - 2vu_{\eta\xi} + u_{\eta\eta}. \end{aligned}$$

Thus $u_{xx} = u_{tt}$ translates as $c^2 u_{\xi\xi} = v^2 u_{\xi\xi} - 2vu_{\eta\xi} + u_{\eta\eta}$.

(ii) Solve the following IVP.

PDE :	$u_{tt}(x, t) - 2u_{tx} = 24u_{xx}(x, t)$	for $-\infty < x < \infty$ and $t > 0$
IC :	$u(x, 0) = 0$	for $-\infty < x < \infty$ and $t > 0$
	$u_t(x, 0) = \sin(x)$	for $-\infty < x < \infty$ and $t > 0$

Solution:
Looking at the solution of the problem before, we realize that it looks oddly similar to the PDE of this exercise. Indeed, observing that $24 = 5^2 - 1^2$, we write the PDE as follows:

$$5^2 u_{xx} = u_{xx} - 2u_{xt} + u_{tt}.$$

Thus, this PDE models the vibration of an infinite string *from the perspective of an observer moving at constant speed.* So, what we need to do to solve the problem is to change the variable in order to get to the reference system of the string (and not of the observer), solve the problem there and then translate it back in the observer's perspective.

The change of variables needs to be opposite, so we get $\xi = x + t$ and $\eta = t$. Computing the derivatives as before we get $u_t = u_\xi + u_\eta$, $u_{xx} = u_{\xi\xi}$, $u_{tt} = u_{\xi\xi} + 2u_{\eta\xi} + u\eta\eta$ and $u_{xt} = u_{\xi\xi} + u_{\xi\eta}$. Thus $5^2 u_{xx} = u_{xx} - 2u_{xt} + u_{tt}$ translates as $5^2 u_{\xi\xi} = u_{\xi\xi} - 2(u_{\xi\xi} = u_{\xi\eta}) + u_{\xi\xi} + 2u_{\eta\xi} + u_{\eta\eta}$, that is $5^2 u_{\xi\xi} = u_{\eta\eta}$.

Let v be a solution of the IVP with the new coordinates, that is:

$$\begin{aligned}
\text{PDE}: \quad & v_{\eta\eta}(\xi,\eta) = 5^2 v_{\xi\xi}(\xi,\eta) && \text{for } -\infty < \xi < \infty \text{ and } \eta > 0 \\
\text{IC}: \quad & v(\xi,0) = 0 && \text{for } -\infty < \xi < \infty \text{ and } \eta > 0 \\
& v_\eta(\xi,0) = \sin(\xi) && \text{for } -\infty < \xi < \infty \text{ and } \eta > 0
\end{aligned}$$

Note that we obtained the IC as follows: since $\xi = x + t$ and $\eta = t$, we have that the initial condition $u(x,0) = 0$ translates as $v(x+0,0) = 0$, that is $v(\xi,0) = 0$. Similarly, $u_t(x,0) = \sin(x)$ translates as $v_\xi(\xi+0,0) + v_\eta(\xi+0,0) = \sin(\xi+0)$. Since the function $v(\xi,0)$ is constant in the variable ξ, its derivative with respect to ξ must be zero. Thus $u_t(x,0) = \sin(x)$ translates as $v_\eta(\xi,0) = \sin(\xi)$.

We can explicitly compute v using d'Alembert:

$$v(\xi,\eta) = \frac{1}{10}\int_{\xi-5\eta}^{\xi+5\eta} \sin(s)ds = \frac{1}{10}\left(\cos(\xi+5\eta) - \cos(\xi-5\eta)\right).$$

Thus the original solution of the IVP is

$$\begin{aligned}
u(x,t) = v(x+t,t) &= \frac{1}{10}\left(\cos((x+t)+5t) - \cos((x+t)-5t)\right) = \\
&= \frac{1}{10}(\cos(x+6t) - \cos(x-4t)).
\end{aligned}$$