Analysis III (BAUG) **Assignment 7**

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Due 15th November 2018

The first 5 Questions of this exercise sheet present some review material on the main IBVPs and IVPs of this course. You should do as many as you think is enough to be confident on those topics. Questions 6 and 7 are easy and are supposed to furnish some intuition on the properties of the wave equation. You should try to solve them. Question 8 explains why the method of characteristics works for an easy case. It consists of three parts, you should try to do at least the first one.

Question 1

For each of the IBVPs below, suppose

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left(b_n \sin(2nt) + c_n \cos(2nt)\right)$$

is a solution, for some constants $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$.

(i)

$$\begin{aligned}
\text{PDE}: &\quad u_{tt}(x,t) = 4u_{xx}(x,t) && \text{for } 0 < x < \pi \text{ and } t > 0\\
\text{BC}: &\quad u(0,t) = 0,\ \ u(\pi,t) = 0 && \text{for } t \geq 0\\
\text{IC}: &\quad u(x,0) = x(\pi - x) && \text{for } 0 \leq x \leq \pi\\
&\quad u_t(x,0) = 1 && \text{for } 0 \leq x \leq \pi
\end{aligned}$$

Solve this IBVP. Decide which of the following are true. (Check all that apply)

- ☐ $b_2 = 0$
- ☐ $b_3 = \frac{4}{3\pi}$
- ☐ $c_2 = \frac{1}{\pi}$
- ☐ $c_3 = \frac{8}{27\pi}$

(ii)

$$\begin{aligned}
\text{PDE}: &\quad u_{tt}(x,t) = 4u_{xx}(x,t) && \text{for } 0 < x < \pi \text{ and } t > 0\\
\text{BC}: &\quad u(0,t) = 0,\ \ u(\pi,t) = 0 && \text{for } t \geq 0\\
\text{IC}: &\quad u(x,0) = 0 && \text{for } 0 \leq x \leq \pi\\
&\quad u_t(x,0) = x^2 && \text{for } 0 \leq x \leq \pi
\end{aligned}$$

Solve this IBVP. Decide which of the following are true. (Check all that apply)

- ☐ $b_2 = -\frac{\pi}{4}$
- ☐ $b_3 = \frac{2\pi}{3} - \frac{8}{27\pi}$
- ☐ $c_2 = 0$
- ☐ $c_3 = \frac{1}{\pi}$

Question 2

[Exam question, 2013]

Let $u(x,t)$ be the solution of the one-dimensional wave equation:

$$\begin{aligned} \text{PDE}: &\quad u_{tt}(x,t) = u_{xx}(x,t) && \text{for } x \in \mathbb{R} \text{ and } t > 0 \\ \text{IC}: &\quad u(x,0) = \begin{cases} 1 & \text{for } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \\ &\quad u_t(x,0) = \begin{cases} 1 & \text{for } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(i) The value of $u\left(0, \frac{1}{2}\right)$ is:

- ☐ 0
- ☐ $\frac{1}{2}$
- ☐ 1
- ☐ $\frac{3}{2}$

(ii) For $x \in \mathbb{R}$ the limit $\lim_{t\to\infty} u(x,t)$ is:

- ☐ 0
- ☐ 1
- ☐ $\frac{1}{2}$
- ☐ 2

Question 3

Let $u(x,t)$ be the solution of the one-dimensional wave equation:

$$\begin{aligned} \text{PDE}: &\quad u_{tt}(x,t) = 4u_{xx}(x,t) && \text{for } x \in \mathbb{R} \text{ and } t > 0 \\ \text{IC}: &\quad u(x,0) = \frac{x}{x^2+1} && \text{for } x \in \mathbb{R} \\ &\quad u_t(x,0) = \begin{cases} \cos(\pi x) & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(i) The value of $u\left(0, \frac{1}{4}\right)$ is:

- ☐ $\frac{1}{2\pi}$
- ☐ 0
- ☐ $\frac{1}{4\pi}$
- ☐ $\frac{4}{17}$

(ii) For $x \in \mathbb{R}$ the limit $\lim_{t\to\infty} u(x,t)$ is:

- ☐ 0
- ☐ $\frac{1}{4\pi}$
- ☐ 1
- ☐ $\frac{1}{2}$

Question 4

[Exam question, 2015]

Use d'Alembert's formula for the following problem about the wave equation on an infinite string.

$$\begin{aligned} \text{PDE}: \quad & u_{tt}(x,t) = u_{xx}(x,t) && \text{for } -\infty < x < \infty \text{ and } t > 0 \\ \text{IC}: \quad & u(x,0) = \begin{cases} 10x - x^2 & \text{for } 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases} \\ & u_t(x,0) = \begin{cases} 2 & \text{for } 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(i) Compute $u(2,2)$ and $u(10,1)$.

(ii) What is the solution $u(x,t)$ in the region $\{(x,t) : x \ge t \ge 0\}$?

Question 5

[Exam question, 2013]

Use d'Alembert's formula for the following problem about the wave equation on an infinite string.

$$\begin{aligned} \text{PDE}: \quad & u_{tt}(x,t) = u_{xx}(x,t) && \text{for } -\infty < x < \infty \text{ and } t > 0 \\ \text{IC}: \quad & u(x,0) = \begin{cases} 8x - 2x^2 & \text{for } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases} \\ & u_t(x,0) = \begin{cases} 16 & \text{for } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

What is the solution $u(x,t)$ in the region $\{(x,t) : x \geq t \geq 0\}$?

Question 6

You want to impress your friend by predicting the behavior of an infinite vibrating string, and you say to them that you can predict the position of the point $x_0 = 0$ at any given time just looking at the initial state of the string. Knowing the d'Alambert's formula, and having carefully computed the value c for which the string satisfies $u_{tt} = c^2 u_{xx}$, you think that this is no big deal. However, you realize afterwards that you will be able to observe the initial conditions of the string only in an interval $[-L, L]$, and the conditions outside that interval will be mystery for you.

(i) For which times t will you still be able to predict the behavior of the string and (probably) impress your friends?

(ii) For which times t, will you be able to tell the solution for another point x_1?

(iii) For which portion of the string can you tell its behavior at a given time t_0?

Question 7

A famous musician will play a concert on an instrument that consists of an infinite string and the audience is sitting around the string. The string has a propagation coefficient $c = 5$ m/s and the musician is playing at the coordinate $x = 0$.

If the musician starts to play at 11:00 a.m., when will a person that is sitting 148.5 Km away from the musician see the string vibrating for the first time?

Question 8

The goal of this exercise is to give a proof to the procedure to solve a hyperbolic equation described in the lecture. This exercise consists of three parts. The first is easy and you should try to do. The second to are a bit harder and not compulsory. However, they are interesting.

In this exercise, we will use the, so called, *operators.*

Small review on operators: Even if the name is scary, those are nothing more than "a partial derivative where the numerator is not specified". For instance the operator $(\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y})$ applied to $u = x + y$ yields 3 as a result, whereas applied to $u = \sin(x)$ yields $\cos(x)$ as a result.

It is important to realize that $\frac{\partial}{\partial x}$ is an operator and $\frac{\partial \xi}{\partial x}$, for instance, is not. The difference is that we can compute the value of $\frac{\partial \xi}{\partial x}$, and this value can be either a constant or a function. However, we cannot compute the value of an operator. Moreover, if the variables x and y are independent, there are no "relations" between operators. This means that if we obtain something like $a\frac{\partial}{\partial x} + \frac{\partial \xi}{\partial y}\frac{\partial}{\partial y} = 0$, this necessary means $a = \frac{\partial \xi}{\partial y} = 0$. If x and y are coordinates, then the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are independent.

Finally, let $(\frac{\partial}{\partial x} + A(x,y)\frac{\partial}{\partial y})$ and $(\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y})$ be two operators. We remind that their composition is $\frac{\partial}{\partial x}(\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}) + A(x,y)\frac{\partial}{\partial y}(\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y})$.

1. Consider the PDE $A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} = 0$, and assume by simplicity that A, B, C are never zero. We can write that as $\left(A(x,y)\frac{\partial^2}{\partial x^2} + B(x,y)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + C(x,y)\frac{\partial^2}{\partial y^2}u = 0\right)$. Show that there are three operators P, Q, V of degree one such that the above can be written as $(P \circ Q + V)u = 0$. *(Hint: The second degree terms will arise from the composition $P \circ Q$. Use the term $+V$ to get rid of unwanted first degree terms.)*

2. Let $C(x,y)$ be a function, and consider the ODE $f'(x) = C(x, f(x))$. Let $\xi(x,y)$ be a function such that for every solution f of the ODE, it holds $\xi(x, f(x)) =$ constant. Show that
$$\xi_x + C(x,y)\xi_y = 0.$$

3. Consider the PDE $u_{xx} - C^2(x,y)u_{yy} = 0$, and assume that C is always non zero. Let ξ and η be the new variables obtained with the method of characteristics. Show that with the new variables, the above PDE looks like
$$u_{\xi\eta}(\xi,\eta) = F(\xi,\eta,u,u_\xi,u_\eta)$$
for some function F. *Note: This is a statement only about the second derivatives*