Analysis III (BAUG) **Assignment 7**

Prof. Dr. Alessandro Sisto Due 15th November 2018

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The first 5 Questions of this exercise sheet present some review material on the main IBVPs and IVPs of this course. You should do as many as you think is enough to be confident on those topics. Questions 6 and 7 are easy and are supposed to furnish some intuition on the properties of the wave equation. You should try to solve them. Question 8 explains why the method of characteristics works for an easy case. It consists of three parts, you should try to do at least the first one.

Question 1

For each of the IBVPs below, suppose

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left(b_n \sin(2nt) + c_n \cos(2nt)\right)$$

is a solution, for some constants $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$.

(i)

$$\begin{aligned}
\text{PDE}: &\quad u_{tt}(x,t) = 4u_{xx}(x,t) &&\text{for } 0 < x < \pi \text{ and } t > 0\\
\text{BC}: &\quad u(0,t) = 0,\ \ u(\pi,t) = 0 &&\text{for } t \geq 0\\
\text{IC}: &\quad u(x,0) = x(\pi - x) &&\text{for } 0 \leq x \leq \pi\\
&\quad u_t(x,0) = 1 &&\text{for } 0 \leq x \leq \pi
\end{aligned}$$

Solve this IBVP. Decide which of the following are true. (Check all that apply)

- ☐ $b_2 = 0$
- ☐ $b_3 = \frac{4}{3\pi}$
- ☐ $c_2 = \frac{1}{\pi}$
- ☐ $c_3 = \frac{8}{27\pi}$

Solution:
Here we have $L = \pi$ and $c^2 = 4$ and so we know from the lectures that the solution to the IBVP above is of the form

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left(b_n \sin(2nt) + c_n \cos(2nt)\right)$$

where $b_n = \frac{1}{2n} b'_n$, $c_n = c'_n$ and b'_n, c'_n are such that

$$u(x,0) = \sum_{n=1}^{\infty} c'_n \sin(nx)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} b'_n \sin(nx)$$

for $x \in (0,\pi)$. In Assignments 4 and 5 these sine Fourier series have already been computed and there we saw that

$$b'_n = \frac{2}{\pi n}(1-(-1)^n) \Leftrightarrow b_n = \frac{1}{\pi n^2}(1-(-1)^n)$$

and

$$c'_n = \frac{4}{\pi n^3}(1-(-1)^n) \Leftrightarrow c_n = \frac{4}{\pi n^3}(1-(-1)^n)$$

Hence, we conclude that

$$u(x,t) = \sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{\pi n^2} \sin(nx) \left(\sin(2nt) + \frac{4}{n}\cos(2nt) \right)$$

(ii)

$$\begin{aligned} \text{PDE}: &\quad u_{tt}(x,t) = 4u_{xx}(x,t) && \text{for } 0 < x < \pi \text{ and } t > 0 \\ \text{BC}: &\quad u(0,t) = 0, \;\; u(\pi,t) = 0 && \text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = 0 && \text{for } 0 \leq x \leq \pi \\ &\quad u_t(x,0) = x^2 && \text{for } 0 \leq x \leq \pi \end{aligned}$$

Solve this IBVP. Decide which of the following are true. (Check all that apply)

- ☐ $b_2 = -\frac{\pi}{4}$
- ☐ $b_3 = \frac{2\pi}{3} - \frac{8}{27\pi}$
- ☐ $c_2 = 0$
- ☐ $c_3 = \frac{1}{\pi}$

Solution:

Here we have $L = \pi$ and $c^2 = 4$ and so we know from the lectures that the solution to the IBVP above is of the form

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left(b_n \sin(2nt) + c_n \cos(2nt)\right)$$

where $b_n = \frac{1}{2n} b'_n$, $c_n = c'_n$ and b'_n, c'_n are such that

$$u(x,0) = \sum_{n=1}^{\infty} c'_n \sin(nx)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} b'_n \sin(nx)$$

for $x \in (0,\pi)$. Clearly, we have $c_n = c'_n = 0$ for all $n \geq 1$ since $u(x,0) \equiv 0$. Moreover the sine Fourier series of $u_t(x,0) = x^2$ for $x \in (0,\pi)$ is indicated in exercise 2, where we see that

$$b'_n = -\frac{2\pi(-1)^n}{n} + \frac{4(-1+(-1)^n)}{n^3\pi} \Leftrightarrow b_n = -\frac{\pi(-1)^n}{n^2} + \frac{2(-1+(-1)^n)}{n^4\pi}$$

Hence, we conclude that

$$u(x,t) = \sum_{n=1}^{\infty} \left(-\frac{\pi(-1)^n}{2n^2} + \frac{2(-1+(-1)^n)}{n^4\pi}\right) \sin(nx)\sin(2nt)$$

Question 2

[Exam question, 2013]

Let $u(x,t)$ be the solution of the one-dimensional wave equation:

$$\begin{aligned}
\text{PDE}: \quad & u_{tt}(x,t) = u_{xx}(x,t) && \text{for } x \in \mathbb{R} \text{ and } t > 0 \\
\text{IC}: \quad & u(x,0) = \begin{cases} 1 & \text{for } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \\
& u_t(x,0) = \begin{cases} 1 & \text{for } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

(i) The value of $u\left(0, \frac{1}{2}\right)$ is:

☐ 0
☐ $\frac{1}{2}$
☐ 1
☐ $\frac{3}{2}$

Solution:
Using d'Alembert's formula we see that

$$u\left(0, \frac{1}{2}\right) = \frac{u\left(-\frac{1}{2}, 0\right) + u\left(\frac{1}{2}, 0\right)}{2} + \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} u_t(x, 0)\, dx = \frac{1+1}{2} + \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} 1\, dx = 1 + \frac{1}{2} = \frac{3}{2}$$

(ii) For $x \in \mathbb{R}$ the limit $\lim_{t\to\infty} u(x, t)$ is:

☐ 0
☐ 1
☐ $\frac{1}{2}$
☐ 2

Solution:
Note that for fixed $x \in \mathbb{R}$ we have $\lim_{t\to\infty} u(x - 2t, 0) = \lim_{s\to\infty} u(-s, 0) = 0$, that $\lim_{t\to\infty} u(x + 2t, 0) = \lim_{s\to\infty} u(s, 0) = 0$ and that $\lim_{t\to\infty} \int_{x-2t}^{x+2t} u_t(s, 0)\, ds = \int_{-1}^{1} 1\, ds = 2$.

Thus, using d'Alembert's formula we see that

$$\lim_{t\to\infty} u(x, t) = \frac{0+0}{2} + \frac{1}{2} \times 2 = 1$$

Question 3

Let $u(x, t)$ be the solution of the one-dimensional wave equation:

$$\begin{aligned}
\text{PDE}: \quad & u_{tt}(x,t) = 4u_{xx}(x,t) && \text{for } x \in \mathbb{R} \text{ and } t > 0 \\
\text{IC}: \quad & u(x,0) = \frac{x}{x^2+1} && \text{for } x \in \mathbb{R} \\
& u_t(x,0) = \begin{cases} \cos(\pi x) & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

(i) The value of $u\left(0, \frac{1}{4}\right)$ is:

- ☐ $\frac{1}{2\pi}$
- ☐ 0
- ☐ $\frac{1}{4\pi}$
- ☐ $\frac{4}{17}$

Solution:
Using d'Alembert's formula we see that

$$u\left(0, \frac{1}{4}\right) = \frac{u\left(-\frac{1}{2}, 0\right) + u\left(\frac{1}{2}, 0\right)}{2} + \frac{1}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}} u_t(x, 0)\, dx$$
$$= \frac{-\frac{2}{5} + \frac{2}{5}}{2} + \frac{1}{4}\int_0^{\frac{1}{2}} \cos(\pi x)\, dx = 0 + \frac{1}{4}\left[\frac{1}{\pi}\sin(\pi x)\right]_0^{\frac{1}{2}} = \frac{1}{4\pi}$$

(ii) For $x \in \mathbb{R}$ the limit $\lim_{t\to\infty} u(x, t)$ is:

- ☐ 0
- ☐ $\frac{1}{4\pi}$
- ☐ 1
- ☐ $\frac{1}{2}$

Solution:
Note that for fixed $x \in \mathbb{R}$ we have $\lim_{t\to\infty} u(x - 2t, 0) = \lim_{s\to\infty} u(-s, 0) = 0$, that $\lim_{t\to\infty} u(x + 2t, 0) = \lim_{s\to\infty} u(s, 0) = 0$ and that $\lim_{t\to\infty} \int_{x-2t}^{x+2t} u_t(s, 0)\, ds = \int_0^1 \cos(\pi s)\, ds = 0$.

Thus, using d'Alembert's formula we see that

$$\lim_{t\to\infty} u(x, t) = \frac{0 + 0}{2} + \frac{1}{4} \times 0 = 0$$

Question 4

[Exam question, 2015]

Use d'Alembert's formula for the following problem about the wave equation on an infinite string.

$$\begin{aligned}
\text{PDE}: \quad & u_{tt}(x,t) = u_{xx}(x,t) && \text{for } -\infty < x < \infty \text{ and } t > 0 \\
\text{IC}: \quad & u(x,0) = \begin{cases} 10x - x^2 & \text{for } 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases} \\
& u_t(x,0) = \begin{cases} 2 & \text{for } 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

(i) Compute $u(2,2)$ and $u(10,1)$.

Solution:
Using d'Alembert's formula we see that

$$u\,(2,2) = \frac{u\,(0,0) + u\,(4,0)}{2} + \frac{1}{2}\int_0^4 u_t(x,0)\,dx = \frac{0+24}{2} + \frac{1}{2}\int_0^4 2\,dx = 12 + 4 = 16$$

and

$$u\,(10,1) = \frac{u\,(9,0) + u\,(11,0)}{2} + \frac{1}{2}\int_9^{11} u_t(x,0)\,dx = \frac{9+0}{2} + \frac{1}{2}\int_9^{10} 2\,dx = \frac{9}{2} + 1 = \frac{11}{2}$$

(ii) What is the solution $u(x,t)$ in the region $\{(x,t) : x \ge t \ge 0\}$?

Solution:
We have three cases to consider:

(a) $x + t \ge x - t \ge 10$: In this case it follows from d'Alembert's formula that

$$u(x,t) = 0$$

(b) $x + t \ge 10 \ge x - t \ge 0$: In this case it follows from d'Alembert's formula that

$$u(x,t) = \frac{10(x-t) - (x-t)^2 + 0}{2} + \frac{1}{2}\int_{x-t}^{10} 2\,ds = 4x - 4t + 10 + xt - \frac{x^2}{2} - \frac{t^2}{2}$$

(c) $10 \geq x+t \geq x-t \geq 0$: In this case it follows from d'Alembert's formula that

$$u(x,t) = \frac{10(x-t)-(x-t)^2+10(x+t)-(x+t)^2}{2} + \frac{1}{2}\int_{x-t}^{x+t} 2\,ds = 10x - x^2 - t^2 + 2t$$

Question 5

[Exam question, 2013]

Use d'Alembert's formula for the following problem about the wave equation on an infinite string.

$$\begin{aligned} \text{PDE}: \quad & u_{tt}(x,t) = u_{xx}(x,t) && \text{for } -\infty < x < \infty \text{ and } t > 0 \\ \text{IC}: \quad & u(x,0) = \begin{cases} 8x - 2x^2 & \text{for } 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases} \\ & u_t(x,0) = \begin{cases} 16 & \text{for } 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

What is the solution $u(x,t)$ in the region $\{(x,t) : x \geq t \geq 0\}$?

Solution:
We have three cases to consider:

1. $x+t \geq x-t \geq 4$: In this case it follows from d'Alembert's formula that

$$u(x,t) = 0$$

2. $x+t \geq 4 \geq x-t \geq 0$: In this case it follows from d'Alembert's formula that

$$u(x,t) = \frac{8(x-t)-2(x-t)^2+0}{2} + \frac{1}{2}\int_{x-t}^{4} 16\,ds = -4x + 4t + 32 + 2xt - x^2 - t^2$$

3. $4 \geq x+t \geq x-t \geq 0$: In this case it follows from d'Alembert's formula that

$$u(x,t) = \frac{8(x-t)-2(x-t)^2+8(x+t)-2(x+t)^2}{2} + \frac{1}{2}\int_{x-t}^{x+t} 16\,ds = 8x - 2x^2 - 2t^2 + 16t$$

Question 6

You want to impress your friend by predicting the behavior of an infinite vibrating string, and you say to them that you can predict the position of the point $x_0 = 0$ at any given time just looking at the initial state of the string. Knowing the d'Alambert's formula, and having carefully computed the value c for which the string satisfies $u_{tt} = c^2 u_{xx}$, you think that this is no big deal. However, you realize afterwards that you will be able to observe the initial conditions of the string only in an interval $[-L, L]$, and the conditions outside that interval will be mystery for you.

(i) For which times t will you still be able to predict the behavior of the string and (probably) impress your friends?

(ii) For which times t, will you be able to tell the solution for another point x_1?

(iii) For which portion of the string can you tell its behavior at a given time t_0?

Solution:
All the questions above can be easily answered using the *domain of dependence*, that was introduced in the lecture. Indeed, the behavior of the point x at time t (that is, the value $u(x,t)$) is completely determined by the values of the initial conditions $f(x)$ and $g(x)$ on the interval $[x - ct, x + ct]$. In particular, this means that you can predict the behavior of $x = 0$ as long as the time t is such that the interval $[0 - ct, 0 + ct]$ is contained in $[-L, L]$, which is the portion of the string where you know the values $f(x)$ and $g(x)$. That is, whenever $t \leq \frac{L}{c}$.

Similarly, for a given point x_1, you can predict its behavior for all t such that $[x_1 - ct, x_1 + ct] \subseteq [-L, L]$, that is, whenever $t \leq \frac{L+|x_1|}{c}$. In particular, if $x_1 \notin [-L, L]$, nothing can be predicted.

Finally, at time t_0 you can predict the behavior of all x such that $t_0 \leq \frac{L-|x|}{c}$, that is, all x in the interval $[-(L - ct_0), L - ct_0]$.

Question 7

A famous musician will play a concert on an instrument that consists of an infinite string and the audience is sitting around the string. The string has a propagation coefficient $c = 5$ m/s and the musician is playing at the coordinate $x = 0$.

If the musician starts to play at 11:00 a.m., when will a person that is sitting 148.5 Km away from the musician see the string vibrating for the first time?

Solution:
First we convert Km to m to match c. Thus we have that the person we are interested in is sit at coordinate $x_1 = 148.500$. We want to find the minimal t_1 such that (x_1, t_1) is

in the *region of influence* of $x_0 = 0$. That is, the minimal t_1 such that $x_1 - ct_1 \geq x_0 = 0$. That is, $t_1 = 29.700$. Since we are measuring in seconds, we have that t_1 is equal to 495 minutes, that are 8 hours and a quarter. Thus this person will see the string vibrating for the first time at 5:15 p.m.

Question 8

The goal of this exercise is to give a proof to the procedure to solve a hyperbolic equation described in the lecture. This exercise consists of three parts. The first is easy and you should try to do. The second to are a bit harder and not compulsory. However, they are interesting.

In this exercise, we will use the, so called, *operators*.

Small review on operators: Even if the name is scary, those are nothing more than "a partial derivative where the numerator is not specified". For instance the operator $(\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y})$ applied to $u = x + y$ yields 3 as a result, whereas applied to $u = \sin(x)$ yields $\cos(x)$ as a result.

It is important to realize that $\frac{\partial}{\partial x}$ is an operator and $\frac{\partial \xi}{\partial x}$, for instance, is not. The difference is that we can compute the value of $\frac{\partial \xi}{\partial x}$, and this value can be either a constant or a function. However, we cannot compute the value of an operator. Moreover, if the variables x and y are independent, there are no "relations" between operators. This means that if we obtain something like $a\frac{\partial}{\partial x} + \frac{\partial \xi}{\partial y}\frac{\partial}{\partial y} = 0$, this necessary means $a = \frac{\partial \xi}{\partial y} = 0$. If x and y are coordinates, then the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are independent.

Finally, let $(\frac{\partial}{\partial x} + A(x,y)\frac{\partial}{\partial y})$ and $(\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y})$ be two operators. We remind that their composition is $\frac{\partial}{\partial x}(\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}) + A(x,y)\frac{\partial}{\partial y}(\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y})$.

1. Consider the PDE $A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} = 0$, and assume by simplicity that A, B, C are never zero. We can write that as $\left(A(x,y)\frac{\partial^2}{\partial x^2} + B(x,y)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + C(x,y)\frac{\partial^2}{\partial y^2}u = 0\right)$. Show that there are three operators P, Q, V of degree one such that the above can be written as $(P \circ Q + V)u = 0$. *(Hint: The second degree terms will arise from the composition $P \circ Q$. Use the term $+V$ to get rid of unwanted first degree terms.)*

Solution:
Treating the PDE as a polynomial, that is $Az^2 + Bzw + Cw^2$, we can find functions a, b, c, d such that $(az + bw)(cz + dw) = Az^2 + Bzw + Cw^2$. Now consider the

following composition:

$$\left(a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}\right)\circ\left(c\frac{\partial}{\partial x}+d\frac{\partial}{\partial y}\right)=a\frac{\partial}{\partial x}\left(c\frac{\partial}{\partial x}+d\frac{\partial}{\partial y}\right)+b\frac{\partial}{\partial y}\left(c\frac{\partial}{\partial x}+d\frac{\partial}{\partial y}\right)=$$

$$=ac_x\frac{\partial}{\partial x}+ac\frac{\partial^2}{\partial x^2}+ad_x\frac{\partial}{\partial y}+ad\frac{\partial}{\partial x}\frac{\partial}{\partial y}+$$

$$+bc_y\frac{\partial}{\partial x}+bc\frac{\partial}{\partial x}\frac{\partial}{\partial y}+bd_y\frac{\partial}{\partial y}+bd\frac{\partial^2}{\partial y^2}=$$

$$=A\frac{\partial^2}{\partial x^2}+B\frac{\partial}{\partial x}\frac{\partial}{\partial y}+C\frac{\partial^2}{\partial y^2}+$$

$$+(ac_x+bc_y)\frac{\partial}{\partial x}+(ad_x+bd_y)\frac{\partial}{\partial y}$$

Letting $P=(a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y})$, $Q=(c\frac{\partial}{\partial x}+d\frac{\partial}{\partial y})$ and $V=-(ac_x+bc_y)\frac{\partial}{\partial x}-(ad_x+bd_y)\frac{\partial}{\partial y}$, we obtain the result.

2. Let $C(x,y)$ be a function, and consider the ODE $f'(x)=C(x,f(x))$. Let $\xi(x,y)$ be a function such that for every solution f of the ODE, it holds $\xi(x,f(x))=$ constant. Show that

$$\xi_x+C(x,y)\xi_y=0.$$

Solution:
Fix a point (x_0,y_0). We will show that $\xi_x(x_0,y_0)+C(x_0,y_0)\xi_y(x_0,y_0)=0$. Since the functions ξ and C are fixed, and this holds for every (x_0,y_0) we get the result.

Let $\overline{f}$ be a solution of $f'(x)=C(x,f(x))$ such that $\overline{f}(x_0)=y_0$. We know from the theory of ODE that such solution exists (and is unique). Then we have $\xi(x,\overline{f}(x))=$ constant for every x. Note that $\xi(x,\overline{f}(x))$ is a function of the only variable x which is constant. Thus, its derivative is zero. Applying the chain rule we get:

$$0=\frac{d}{dx}\xi(x,\overline{f}(x))=\xi_x+\frac{d\overline{f}}{dx}\xi_y=\xi_x+C(x,\overline{f}(X))\xi_y.$$

However, for the value $x=x_0$, we get that $y_0=\overline{f}(x_0)$. Thus, we obtain the desired result. Note that it is important that $\xi(x,f(x))$ is constant for *all* solutions of $f'(x)=C(x,f(x))$.

3. Consider the PDE $u_{xx}-C^2(x,y)u_{yy}=0$, and assume that C is always non zero. Let ξ and η be the new variables obtained with the method of characteristics. Show that with the new variables, the above PDE looks like

$$u_{\xi\eta}(\xi,\eta)=F(\xi,\eta,u,u_\xi,u_\eta)$$

for some function F. *Note: This is a statement only about the second derivatives*

Solution:
Using the methods of characteristics, we consider the ODE $(f'(x))^2 = C^2(x, f(x))$, which admits two family of solutions, namely the functions f_1 that satisfy $f_1(x) = C(x, f_2(x))$, and those which satisfy $f_2(x) = -C(x, f_2(x))$. We find ξ and η such that for each member of the first family, we have $\xi(x, f_1(x)) =$ constant, and similarly for η. In particular, the second point gives us that the following equations holds:

$$\xi_x + C\xi_y = 0$$
$$\eta_x - C\eta_y = 0$$

Decomposing the PDE $u_{xx} - C^2(x, y)u_{yy} = 0$ as $(P \circ Q + V)u = 0$ as in part (i), we get $P = (\frac{\partial}{\partial x} + C\frac{\partial}{\partial y})$ and $Q = (\frac{\partial}{\partial x} - C\frac{\partial}{\partial y})$. Note that all terms of degree 2 are produced by the composition $P \circ Q$. Since the operator V will produce only elements of degree 1, which we incorporate in the function F of the statement, we can basically ignore V. Changing variables, we obtain:

$$P \circ Q = (\xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta} + C\xi_y \frac{\partial}{\partial \xi} + C\eta_y \frac{\partial}{\partial \eta}) \circ (\xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta} - C\xi_y \frac{\partial}{\partial \xi} - C\eta_y \frac{\partial}{\partial \eta}).$$

However, from the two equations above, we we get:

$$P \circ Q = \left((\eta_x + C\eta_y) \frac{\partial}{\partial \eta} \right) \circ \left((\xi_x - C\xi_y) \frac{\partial}{\partial \xi} \right)$$

Thus, the only term of second degree is going to be $u_{\xi\eta}$.