Analysis III (BAUG) **Assignment 8**

Prof. Dr. Alessandro Sisto Due 21th November 2018

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The first 6 questions are a review of the topics of the midterm. Question 8 is the hard exercise of this series.

Question 1

Which of the following PDEs are homogeneous? (Check all that apply.)

- ☐ $u_{xx} - 2u_{xy} + u_{yy} = 0$
- ☐ $u_{xx} - 4u_x + \frac{1}{y^2}u - 3\sin(x) = 4\cos(x)$
- ☐ $4u_{xx} = -y^2x^2u_y$
- ☐ $u_{xyx} + u_{xy} + \sin(x)u_{xx} - u_{yy} + e^{x^2}u_x = x^2y$
- ☐ $u_x - e^xy - u_{xy} + \frac{y}{x^2}u = 0$

Question 2

Which of the following PDEs are linear? (Check all that apply.)

- ☐ $u_{xy} - 4u_x + 8u_y = 2\sin(y)$
- ☐ $u_{xt} - u^2 + e^tu_tt = 0$
- ☐ $u_x - e^{-y^2}u_{yx} - u_y = 0$
- ☐ $\frac{1}{x^2+y^2}u_x - 2u_y = 10xyu_xu_y + e^{x+y}$
- ☐ $su_s + s^2 = zu_z + z^2$

Question 3

Consider the following 2nd order linear PDE

$$u_y + (3+x)u_{xx} - 2u_{xy} + (3-x)u_{yy} + \sin(y)u_x = e^{x^2+y^2}$$

This PDE is:

- ☐ Hyperbolic
- ☐ Parabolic
- ☐ Elliptic
- ☐ Of mixed type

Question 4

Consider the following IBVP:

$$\begin{aligned}
\text{PDE}: &\quad u_t(x,t) = 3u_{xx}(x,t) &&\text{for } -\pi < x < \pi \text{ and } t > 0\\
\text{BC}: &\quad u(-\pi,t) = u(\pi,t),\;\; u_x(-\pi,t) = u_x(\pi,t) &&\text{for } t \geq 0\\
\text{IC}: &\quad u(x,0) = 1 + x &&\text{for } -\pi < x < \pi
\end{aligned}$$

1. Which of the following problems does the IBVP above model?

   - ☐ The heat flow on a laterally insulated rod with the temperature fixed at both ends.
   - ☐ The heat flow on a completely insulated rod.
   - ☐ The heat flow on a circular wire.
   - ☐ None of the above.

2. Suppose $u(x,t)$ is a solution to the IBVP above and that

   $$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-c_n t}\cos(nx) + \sum_{n=1}^{\infty} b_n e^{-d_n t}\sin(nx)$$

   for some constants $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 1}$, $\{c_n\}_{n\geq 1}$ and $\{d_n\}_{n\geq 1}$. Which of the following are true?

   - ☐ $a_2 = 1 + (-1)^3$
   - ☐ $a_3 = -\frac{2}{3}$
   - ☐ $b_4 = -\frac{1}{2}$
   - ☐ $b_5 = 1 + \frac{2}{5}$
   - ☐ $d_6 = -39$

Question 5

Consider the following IBVP:

$$\begin{aligned}
\text{PDE}: &\quad u_t(x,t) = 4u_{xx}(x,t) &&\text{for } 0 < x < 2 \text{ and } t > 0\\
\text{BC}: &\quad u_x(0,t) = 0,\;\; u_x(3,t) = 0 &&\text{for } t \geq 0\\
\text{IC}: &\quad u(x,0) = \begin{cases} 1 \text{ for } 0 \leq x \leq 1\\ 2 \text{ for } 1 < x \leq 2\end{cases}
\end{aligned}$$

1. Which of the following problems does the IBVP above model?

   - ☐ The heat flow on a laterally insulated rod with the temperature fixed at both ends.

- ☐ The heat flow on a completely insulated rod.
- ☐ The heat flow on a circular wire.
- ☐ None of the above.

2. Suppose $u(x,t)$ is a solution to the IBVP above and that

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-c_n t} \cos\left(\tfrac{n\pi}{2}x\right) + \sum_{n=1}^{\infty} b_n e^{-d_n t} \sin\left(\tfrac{n\pi}{2}x\right)$$

for some constants $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 1}$, $\{c_n\}_{n\geq 1}$ and $\{d_n\}_{n\geq 1}$. Which of the following are true?

- ☐ $a_0 = \frac{3}{2}$
- ☐ $a_1 = 0$
- ☐ $b_1 = -\frac{2}{\pi}$
- ☐ $b_2 = 0$
- ☐ $c_3 = 6\pi$

Question 6

Consider the following IVP:

$$\begin{aligned} \text{PDE}: &\quad u_{tt}(x,t) = 49u_{xx}(x,t) &&\text{for } -\infty < x < \infty \text{ and } t > 0 \\ \text{IC}: &\quad u(x,0) = \begin{cases} 2 \text{ for } x \in [2,67] \\ 0 \text{ otherwise} \end{cases} \\ &\quad u_t(x,0) = \sin(2\pi x) + \sin\left(\frac{3}{4}\pi x\right) \end{aligned}$$

(a) Which of the following problems does the IVP above model?

- ☐ The heat flow on a laterally insulated rod with the temperature fixed at both ends.
- ☐ The heat flow on a completely insulated rod.
- ☐ The heat flow on a circular wire.
- ☐ None of the above.

(b) Suppose $u(x,t)$ is a solution to the IVP above. Which of the following are true?

- ☐ $u(0,7) = 1 + \frac{2}{7\pi}$
- ☐ $u(1,1) = 1 - \frac{2}{21\pi}$
- ☐ $u(20,2) = 2$
- ☐ $u(0,\frac{1}{4}) = 1$
- ☐ $u(7,3) = 0$

Question 7

[Exercise 5.1 of Lecture 8] To practice using lemma seen in the lecture, check that any polynomial is of exponential order.

(a) Use the lemma to check that any power function t^n is of exponential order.

(b) Use the lemma a second time to check that if $f(t)$ and $g(t)$ are of exponential order, and if α and β are constants, then $\alpha f(t) + \beta g(t)$ is also of exponential order.

(c) Use the previous two facts to deduce that any polynomial is of exponential order.

Question 3

The goal of this question is to prove the uniqueness of the Laplace transform.

1. Let $h(u)$ be a continuous function on $[0, 1]$ and suppose that for each $n = 0, 1, 2, \ldots$ it holds:
$$\int_0^1 h(u)u^n du = 0.$$
Show that $h(u) = 0$ for each u.

   You may want to use the following fact: for each function $h(u)$ and for each $\epsilon > 0$, there is a polynomial $P_\epsilon(u)$ such that $\lim_{\epsilon \to 0} |h(u) - P_\epsilon(u)| < \epsilon$. Also, always assume that all the requirements are satisfied so that limits are "well behaved".

2. Let $f(t)$ be a function defined on $[0, \infty)$. Suppose that $(\mathcal{L}f)(s) = 0$ for each $s > a$, where c is the exponential order of f. Conclude that $f(t) = 0$ for each $t \geq 0$.

   Hint! Use the change of variables $u = e^{-t}$ (don't worry about the behavior at ∞). Moreover, try to understand the situation at a point $s = s_0 + n + 1$, where s_0 is any positive (real) number greater than c.

3. Conclude that if there is c such that for each $s > c$ we have $\mathcal{L}f(s) = \mathcal{L}g(s)$, then $f(t) = g(t)$ for each $t \geq 0$.