Analysis III (BAUG) **Assignment 8**

Prof. Dr. Alessandro Sisto Due 21th November 2018

Organizer: Davide Spriano

The first 6 questions are a review of the topics of the midterm. Question 8 is the hard exercise of this series.

Question 1

Which of the following PDEs are homogeneous? (Check all that apply.)

- ☐ $u_{xx} - 2u_{xy} + u_{yy} = 0$**Yes**
- ☐ $u_{xx} - 4u_x + \frac{1}{y^2}u - 3\sin(x) = 4\cos(x)$
- ☐ $4u_{xx} = -y^2x^2u_y$**Yes**
- ☐ $u_{xyx} + u_{xy} + \sin(x)u_{xx} - u_{yy} + e^{x^2}u_x = x^2y$
- ☐ $u_x - e^x y - u_{xy} + \frac{y}{x^2}u = 0$

Solution:
Number 1 and 3

Question 2

Which of the following PDEs are linear? (Check all that apply.)

- ☐ $u_{xy} - 4u_x + 8u_y = 2\sin(y)$**Yes**
- ☐ $u_{xt} - u^2 + e^t u_t t = 0$
- ☐ $u_x - e^{-y^2}u_{yx} - u_y = 0$**Yes**
- ☐ $\frac{1}{x^2+y^2}u_x - 2u_y = 10xyu_xu_y + e^{x+y}$
- ☐ $su_s + s^2 = zu_z + z^2$**Yes**

Solution:
Number 1, 3 and 5

Question 3

Consider the following 2nd order linear PDE

$$u_y + (3+x)u_{xx} - 2u_{xy} + (3-x)u_{yy} + \sin(y)u_x = e^{x^2+y^2}$$

This PDE is:

- ☐ Hyperbolic
- ☐ Parabolic
- ☐ Elliptic
- ☐ Of mixed type **Yes**

> **Solution:**
> Since the positivity of $4 - 4(9 - x^2)$ depends on x, the PDE is mixed.

Question 4

Consider the following IBVP:

PDE :	$u_t(x,t) = 3u_{xx}(x,t)$	for $-\pi < x < \pi$ and $t > 0$
BC :	$u(-\pi,t) = u(\pi,t), \quad u_x(-\pi,t) = u_x(\pi,t)$	for $t \geq 0$
IC :	$u(x,0) = 1 + x$	for $-\pi < x < \pi$

1. Which of the following problems does the IBVP above model?
   - ☐ The heat flow on a laterally insulated rod with the temperature fixed at both ends.
   - ☐ The heat flow on a completely insulated rod.
   - ☐ The heat flow on a circular wire. **Yes**
   - ☐ None of the above.

> **Solution:**
> It models the heat flow on a circular wire.

2. Suppose $u(x,t)$ is a solution to the IBVP above and that

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-c_n t} \cos(nx) + \sum_{n=1}^{\infty} b_n e^{-d_n t} \sin(nx)$$

for some constants $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 1}$, $\{c_n\}_{n\geq 1}$ and $\{d_n\}_{n\geq 1}$. Which of the following are true?

- ☐ $a_2 = 1 + (-1)^3$**Yes**
- ☐ $a_3 = -\frac{2}{3}$
- ☐ $b_4 = -\frac{1}{2}$**Yes**
- ☐ $b_5 = 1 + \frac{2}{5}$
- ☐ $d_6 = -39$

Solution:
Observe that the Fourier transform of $1 + x$ is the same as the Fourier transform of 1 plus the Fourier transform of x. However, we know that the Fourier transform of 1 is 1, and the Fourier transform of x in the interval $[-\pi, \pi]$ is $\sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nx)$ (see Lecture 4, Example 2.3). Thus we obtain $a_0 = 1, a_n = 0, b_n = (-1)^{n+1}\frac{2}{n}$. Thus we have that the correct answers are 1 and 3.

Question 5

Consider the following IBVP:

$$\begin{aligned} \text{PDE}: \quad & u_t(x,t) = 4u_{xx}(x,t) && \text{for } 0 < x < 2 \text{ and } t > 0 \\ \text{BC}: \quad & u_x(0,t) = 0, \quad u_x(3,t) = 0 && \text{for } t \geq 0 \\ \text{IC}: \quad & u(x,0) = \begin{cases} 1 \text{ for } 0 \leq x \leq 1 \\ 2 \text{ for } 1 < x \leq 2 \end{cases} \end{aligned}$$

1. Which of the following problems does the IBVP above model?
   - ☐ The heat flow on a laterally insulated rod with the temperature fixed at both ends.
   - ☐ The heat flow on a completely insulated rod.**Yes**
   - ☐ The heat flow on a circular wire.
   - ☐ None of the above.

Solution:
A completely insulated rod.

2. Suppose $u(x,t)$ is a solution to the IBVP above and that

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-c_n t}\cos\left(\tfrac{n\pi}{2}x\right) + \sum_{n=1}^{\infty} b_n e^{-d_n t}\sin\left(\tfrac{n\pi}{2}x\right)$$

for some constants $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 1}$, $\{c_n\}_{n\geq 1}$ and $\{d_n\}_{n\geq 1}$. Which of the following are true?

- ☐ $a_0 = \frac{3}{2}$ **Yes**
- ☐ $a_1 = 0$
- ☐ $b_1 = -\frac{2}{\pi}$
- ☐ $b_2 = 0$**Yes**
- ☐ $c_3 = 6\pi$

Solution:
We extend $\phi(x) = u(x,0)$ to its Fourier cosine series:

$$a_0 = \frac{1}{2}\int_0^2 \phi(x)d(x) = \frac{1}{2}\left(\int_0^1 1dx + \int_1^2 2dx\right) = \frac{3}{2}$$
$$a_n = \int_0^2 \phi(x)\cos\left(\frac{n\pi}{2}x\right)dx = \int_0^1 \cos\left(\frac{n\pi}{2}x\right)dx + 2\int_1^2 \cos\left(\frac{n\pi}{2}x\right)dx = -\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

Thus the correct answers are number 1 and 4.

Question 6

Consider the following IVP:

$$\begin{aligned} \text{PDE :} \quad & u_{tt}(x,t) = 49u_{xx}(x,t) && \text{for } -\infty < x < \infty \text{ and } t > 0 \\ \text{IC :} \quad & u(x,0) = \begin{cases} 2 \text{ for } x \in [2,67] \\ 0 \text{ otherwise} \end{cases} \\ & u_t(x,0) = \sin(2\pi x) + \sin\left(\frac{3}{4}\pi x\right) \end{aligned}$$

(a) Which of the following problems does the IVP above model?

- ☐ The heat flow on a laterally insulated rod with the temperature fixed at both ends.
- ☐ The heat flow on a completely insulated rod.
- ☐ The heat flow on a circular wire.
- ☐ None of the above.**Yes**

Solution:
It models the vibration of an infinite string.

(b) Suppose $u(x,t)$ is a solution to the IVP above. Which of the following are true?

- ☐ $u(0,7) = 1 + \frac{2}{7\pi}$
- ☐ $u(1,1) = 1 - \frac{2}{21\pi}$**Yes**
- ☐ $u(20,2) = 2$ **Yes**
- ☐ $u(0, \frac{1}{4}) = 1$
- ☐ $u(7,3) = 0$

Solution:
Second and third are correct

Question 7

[Exercise 5.1 of Lecture 8] To practice using lemma seen in the lecture, check that any polynomial is of exponential order.

(a) Use the lemma to check that any power function t^n is of exponential order.

Solution:
We need to compute $\lim_{t\to\infty} \frac{t^n}{e^{ct}}$. It is a well known fact that this limit is zero for each value of c. We report a proof of this fact.

We proceed by induction on n. If $n = 0$, then the result is clear. If $n > 0$, suppose, by induction, that we know that the limit of $\frac{t^{n-1}}{e^{ct}}$ is zero. Since $\lim_{t\to\infty} t^n = \lim_{t\to\infty} e^{ct} = \infty$, we can apply de l'Hopital. Thus we have:

$$\lim_{t\to\infty} \frac{t^n}{e^{ct}} = \lim_{t\to\infty} \frac{nt^{n-1}}{ce^{ct}} = \frac{n}{c} \cdot 0 = 0.$$

(b) Use the lemma a second time to check that if $f(t)$ and $g(t)$ are of exponential order, and if α and β are constants, then $\alpha f(t) + \beta g(t)$ is also of exponential order.

Solution:
We have that exist c_1, c_2, L_1, L_2 such that $\lim_{t\to\infty} \frac{f(t)}{e^{c_1 t}} = L_1$ and $\lim_{t\to\infty} \frac{g(t)}{e^{c_2 t}} = L_2$. Suppose that $c_2 \geq c_1$ and let c_0 be $c_2 - c_1$. Since the limit of the sum is the sum of the limits, and since $\lim_{t\to\infty} \alpha F(t) = \alpha \lim_{t\to\infty} F(t)$, we get

$$\alpha L_1 + \beta L_2 = \lim_{t\to\infty} \frac{\alpha f(t)}{e^{c_1 t}} + \frac{\beta g(t)}{e^{c_2 t}} = \lim_{t\to\infty} \frac{e^{c_0 t} f(t) + g(t)}{e^{c_2 t}}.$$

So $\alpha f(t) + \beta g(t)$ is of exponential order with constant $c_2 = \max\{c_1, c_2\}$.

(c) Use the previous two facts to deduce that any polynomial is of exponential order.

Solution:
This is because a polynomial is of the form $p(t) = a_n t^n + \cdots + a_0$. Since for each n and $c > 0$, we have $\lim_{t\to\infty} \frac{t^n}{e^{ct}} = 0$, part (b) guarantees that $\lim_{t\to\infty} \frac{p(t)}{e^{ct}} = 0$.

Question 3

The goal of this question is to prove the uniqueness of the Laplace transform.

1. Let $h(u)$ be a continuous function on $[0, 1]$ and suppose that for each $n = 0, 1, 2, \ldots$ it holds:
$$\int_0^1 h(u)u^n du = 0.$$
Show that $h(u) = 0$ for each u.

   You may want to use the following fact: for each function $h(u)$ and for each $\epsilon > 0$, there is a polynomial $P_\epsilon(u)$ such that $\lim_{\epsilon\to 0} |h(u) - P_\epsilon(u)| < \epsilon$. Also, always assume that all the requirements are satisfied so that limits are "well behaved".

Solution:

For each ϵ, we can find $P_\epsilon(u)$ such that $|h(u) - P_\epsilon(u)| < \epsilon$. Since P_ϵ is the sum of terms of the form u^n (it is a polynomial), using the linearity of the integral we get
$$\int_0^1 h(u)P_\epsilon(u)du = 0.$$
Taking the limit for $\epsilon \to 0$, we obtain:
$$\int_0^1 h(u) \cdot h(u)du = 0.$$
However, since $h(u)^2 \geq 0$ for each u, we need to have $h(u) \equiv 0$.

2. Let $f(t)$ be a function defined on $[0, \infty)$. Suppose that $(\mathcal{L}f)(s) = 0$ for each $s > a$, where c is the exponential order of f. Conclude that $f(t) = 0$ for each $t \geq 0$.

   Hint! Use the change of variables $u = e^{-t}$ (don't worry about the behavior at ∞). Moreover, try to understand the situation at a point $s = s_0 + n + 1$, where s_0 is any positive (real) number greater than c.

Solution:
We will use the change of variables $u = e^{-t}$ and let $s_n = s_0 + n + 1$ for $n = 0, 1, 2, \ldots$.

Since s_n is always greater than c, for each $n = 0, 1, 2, \ldots$, we have $\mathcal{L}f(s_n) = 0$. That is

$$0 = \int_0^\infty e^{-s_n t} f(t)dt = \int_0^\infty e^{-nt} e^{-s_0 t} e^{-t} f(t)dt = -\int_0^1 u^n \cdot (u^{s_0} f(-\log(u))du.$$

Setting $h(u) = u^{s_0} f(-\log(u))$, part 1 gives that $h(u) = 0$ for each u. This implies that $f(-\log(u)) = 0$ for each u, thus that $f = 0$ for each t.

3. Conclude that if there is c such that for each $s > c$ we have $\mathcal{L}f(s) = \mathcal{L}g(s)$, then $f(t) = g(t)$ for each $t \geq 0$.

Solution:
By linearity we have that $\mathcal{L}(f - g)(s) = 0$ for each $s > c$. By part 2, we have that $(f - g)(t) = 0$ for each $t \geq 0$. Thus $f(t) = g(t)$ for each $t \geq 0$.