Analysis III (BAUG) **Assignment 9**

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The goal of this exercise sheet is to familiarize with the Laplace transform. There are two types of exercises, grouped in Questions 1 and 2 respectively. Given the importance of the topic, there is no hard question this time: focus on practising the Laplace transform.

Question 1

Find the inverse f of each of the following Laplace transforms F.

(i) $F(s) = \frac{-2}{s+3}$

- ☐ $f(t) = -8$
- ☐ $f(t) = -2e^{-3t}$ **Yes**
- ☐ $f(t) = -2e^{3t}$
- ☐ $f(t) = \frac{-2e^{-3t}}{t}$
- ☐ $f(t) = -6e^{3t}$

Solution:
The Laplace inverse of $\frac{1}{s}$ is 1. By the shifting property (i.e. the fact that $\mathcal{L}\{e^{-at}g(t)\} = G(t+a)$) this means that the inverse of $\frac{1}{s+3}$ is e^{-3t} and hence $f(s) = -2e^{-3t}$

(ii) $F(s) = \frac{3s+13}{s^2+4s+3}$

- ☐ $f(t) = 5e^{t} + 2e^{3t}$
- ☐ $f(t) = 5e^{-t} + 2e^{-3t}$ **Yes**
- ☐ $f(t) = 5e^{t} - 2e^{3t}$
- ☐ $f(t) = 5e^{-t} - 2e^{-3t}$

Solution:
We apply the method of partial fractions. The denominator factors as $s^2+4s+3 = (s+1)(s+3)$, so we want to write $F(s)$ as $\frac{A}{s+1} + \frac{B}{s+3}$.

Putting this over a common denominator, we get

$$3s + 13 = A(s+3) + B(s+1) = (A+B)s + 3A + B,$$

hence, comparing the coefficients, we see $3 = A + B$ and $13 = 3A + B$. Solving this equation (eg by subtracting the first one from the other), we get $A = 5$ and $B = -2$.

So $F(s) = \frac{5}{s+1} + \frac{-2}{s+3}$. The Laplace inverse of the second term is $-2e^{-3t}$ as seen in the previous exercise, and similarly the inverse of the first term is $5e^{-t}$, so $f(t) = 5e^{-t} - 2e^{-3t}$

(iii) $F(s) = \frac{s^2+1}{s^2(s+1)}$

- ☐ $f(t) = 2e^{t} + t^2 - t$
- ☐ $f(t) = 2e^{t} + t - 1$
- ☐ $f(t) = 2e^{-t} + t^2 - t$
- ☐ $f(t) = 2e^{-t} + t - 1$ **Yes**

Solution:
We apply the method of partial fractions: we want to write $F(s)$ as $\frac{As+B}{s^2} + \frac{C}{s+1}$. Bring this to a common denominator, we get that the numerator is

$$s^2 + 1 = (As + B)(s + 1) + Cs^2 = (A + C)s^2 + (A + B)s + B,$$

hence, comparing the coefficients, we see $1 = A + C$, $0 = A + B$ and $1 = B$. Therefore $A = -1$ and $C = 2$.

So $F(s) = \frac{-s+1}{s^2} + \frac{2}{s+1} = \frac{-1}{s} + \frac{1}{s^2} + \frac{2}{s+1}$. The Laplace inverse of the first term is -1, the inverse of the second term is t and the inverse of the third term is $2e^{-t}$ so $f(t) = 2e^{-t} + t - 1$.

(iv) $F(s) = \frac{s+1}{s^2+4}$

- ☐ $f(t) = \cos(2t) + \frac{1}{2}\sin(2t)$ **Yes**
- ☐ $f(t) = \cos(2t) + \sin(2t)$
- ☐ $f(t) = e^{-t}\cos(2t)$
- ☐ $f(t) = e^{t}\cos(2t)$

Solution:
The Laplace inverse of $\frac{s}{s^2+4}$ is $\cos(2t)$ and the Laplace inverse of $\frac{2}{s^2+4}$ is $\sin(2t)$.

Therefore, the Laplace inverse of $\frac{s+1}{s^2+4} = \frac{s}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}$ is $\cos(2t) + \frac{1}{2}\sin(2t)$.

(v) $F(s) = \frac{s^3+s+1}{(s^2+1)(s^2+4)}$

- ☐ $f(t) = \frac{1}{3}\sin(t) + \cos(2t) + \frac{1}{6}\sin(2t)$
- ☐ $f(t) = \frac{1}{3}\sin(t) + \cos(2t) - \frac{1}{6}\sin(2t)$ **Yes**
- ☐ $f(t) = -\frac{1}{3}\sin(t) + \cos(2t) + \frac{1}{6}\sin(2t)$
- ☐ $f(t) = -\frac{1}{3}\sin(t) - \cos(2t) + \frac{1}{6}\sin(2t)$

Solution:
We apply the method of partial fractions: we want to write $F(s)$ as $\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$. Bringing this to a common denominator, we get that the numerator is

$$\begin{aligned} s^3+s+1 &= (As+B)(s^2+4) + (Cs+D)(s^2+1) \\ &= (A+C)s^3 + (B+D)s^2 + (4A+C)s + 4B + D, \end{aligned}$$

hence, comparing the coefficients, we see $1 = A + C$, $0 = B + D$, $1 = 4A + C$ and $1 = 4B + D$. Subtracting the first equation from the third and the second from the fourth, we get $0 = 3A$ and $1 = 3B$. Thus $A = 0$, $B = 1/3$, $C = 1$ and $D = -1/3$.

So $F(s) = \frac{1/3}{s^2+1} + \frac{s-1/3}{s^2+4} = \frac{1/3}{s^2+1} + \frac{s}{s^2+4} + \frac{-1/3}{s^2+4}$. The Laplace inverse of the first term is $\frac{1}{3}\sin(t)$, the inverse of the second term is $\cos(2t)$ and the inverse of the third term is $\frac{-1}{6}\sin(2t)$ so $f(t) = \frac{1}{3}\sin(t) + \cos(2t) - \frac{1}{6}\sin(2t)$.

(vi) $F(s) = \frac{s+1}{(s+1)^2+4}$

- ☐ $f(t) = \cos(2t) + \frac{1}{2}\sin(2t)$
- ☐ $f(t) = \cos(2t) + \sin(2t)$
- ☐ $f(t) = e^{-t}\cos(2t)$ **Yes**
- ☐ $f(t) = e^{t}\cos(2t)$

Solution:
For $G(s) = \frac{s}{s^2+4}$, we know that $g(t) = \cos(2t)$.

But here $F(s) = G(s+1)$, so $f(t) = e^{-t}\cos(2t)$.

(vii) $F(s) = \frac{2s}{s^2+2s+5}$

- ☐ $f(t) = 2e^{-t}\cos(2t) - e^{-t}\sin(2t)$ **Yes**
- ☐ $f(t) = 2e^{-t}\cos(2t) + e^{-t}\sin(2t)$

- ☐ $f(t) = 2e^t \cos(2t) - e^t \sin(2t)$
- ☐ $f(t) = 2e^t \cos(2t) + e^t \sin(2t)$

Solution:
The denominator is irreducible (cannot be written as $(s-a)(s-b)$ with a, b real numbers), so we should write it as a sum of two squares: $(s+1)^2 + 2^2$.

Then

$$F(s) = \frac{2s}{(s+1)^2 + 2^2} = \frac{2(s+1) - 2}{(s+1)^2 + 2^2} = \frac{2(s+1)}{(s+1)^2 + 2^2} + \frac{-2}{(s+1)^2 + 2^2}.$$

Here the previous exercise shows that the inverse of the first term is $2e^{-t}\cos(2t)$, and similarly (again, shifting $\sin(2t)$ by one) we get that the inverse of the second term is $e^{-t}\sin(2t)$. Therefore $f(t) = 2e^{-t}\cos(2t) - e^{-t}\sin(2t)$.

Question 2

Use the Laplace transform to solve the following initial value problems.

(i) $x''(t) + 4x'(t) + 3x(t) = 0, \qquad x(0) = 3, \qquad x'(0) = 1$

Solution:
Applying the Laplace transform, the ODE turns into

$$\mathcal{L}\{x''\} + 4\mathcal{L}\{x'\} + 3\mathcal{L}\{x\} = s^2X(s) - sx(0) - x'(0) + 4[sX(s) - x(0)] + 3X(s) = 0$$

Then we have

$$(s^2 + 4s + 3)X(s) - (s+4)x(0) - x'(0) = (s^2 + 4s + 3)X(s) - [3(s+4) + 1] = 0$$

and hence

$$X(s) = \frac{3(s+4) + 1}{s^2 + 4s + 3} = \frac{3s + 13}{s^2 + 4s + 3}$$

From the previous exercise, we know that the Laplace inverse of this $X(s)$ is $x(t) = 5e^{-t} - 2e^{-3t}$, giving us the solution of the initial value problem.

(ii) $x'(t) + x(t) = t, \qquad x(0) = 1$

Solution:
Applying the Laplace transform, the ODE turns into

$$\mathcal{L}\{x'\} + \mathcal{L}\{x\} = sX(s) - x(0) + X(s) = \frac{1}{s^2} = \mathcal{L}\{t\}$$

Then we have

$$(s+1)X(s) - x(0) = (s+1)X(s) - 1 = \frac{1}{s^2}$$

and hence

$$X(s) = \frac{s^2+1}{s^2(s+1)}$$

From the previous exercise, we know that the Laplace inverse of this $X(s)$ is $x(t) = 2e^{-t} + t - 1$, giving us the solution of the initial value problem. (You can easily substitute it back in the ODE to check that this is indeed the solution!)

(iii) $x''(t) + 4x(t) = \sin(t), \qquad x(0) = 1, \qquad x'(0) = 0$

Solution:
Applying the Laplace transform, the ODE turns into

$$\mathcal{L}\{x''\} + 4\mathcal{L}\{x\} = s^2X(s) - sx(0) - x'(0) + 4X(s) = \frac{1}{s^2+1} = \mathcal{L}\{\sin(t)\}$$

Then we have

$$(s^2+4)X(s) - sx(0) - x'(0) = (s^2+4)X(s) - s = \frac{1}{s^2+1}$$

and hence

$$X(s) = \frac{s^3+s+1}{(s^2+1)(s^2+4)}$$

From the previous exercise, we know that the Laplace inverse of this $X(s)$ is $x(t) = \frac{1}{3}\sin(t) + \cos(2t) - \frac{1}{6}\sin(2t)$, so that is the solution of our initial value problem.

(iv) $x''(t) + 2x'(t) + 5x(t) = 0, \qquad x(0) = 2, \qquad x'(0) = -4$

Solution:
Applying the Laplace transform, the ODE turns into

$$\mathcal{L}\{x''\} + 2\mathcal{L}\{x'\} + 5\mathcal{L}\{x\} = s^2X(s) - sx(0) - x'(0) + 2[sX(s) - x(0)] + 5X(s) = 0$$

Then we have

$$(s^2 + 2s + 5)X(s) - (s + 2)x(0) - x'(0) = (s^2 + 2s + 5)X(s) - [2(s + 2) - 4] = 0$$

and hence

$$X(s) = \frac{2s}{s^2 + 2s + 5}$$

From the previous exercise, we know that the Laplace inverse of this $X(s)$ is $x(t) = 2e^{-t}\cos(2t) - e^{-t}\sin(2t)$, giving us the solution of the initial value problem.