Analysis III (BAUG) **Assignment 1**

Prof. Dr. Alessandro Sisto — Due 27 of September 2018

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Note: No hard exercise for this week.

1. Verify that the each of the following functions satisfy the PDE "$u_{rr} + u_{tt} = 0$".

- $u(r,t) = e^{2r}\cos 2(t)$

Solution:
$u_{rr} + u_{tt} = \frac{\partial^2}{\partial r^2}(e^{2r}\cos(2t)) + \frac{\partial^2}{\partial t^2}(e^{2r}\cos(2t))$
$= \frac{\partial}{\partial r}(2e^{2r}\cos(2t)) + \frac{\partial}{\partial t}(-2e^{2r}\sin(2t))$
$= 4e^{2r}\cos(2t) + (-4e^{2r}\cos(2t)) = 0$

- $u(r,t) = 3r^2t - t^3$

Solution:
$u_{rr} + u_{tt} = \frac{\partial^2}{\partial r^2}(3r^2t - t^3) + \frac{\partial^2}{\partial t^2}(3r^2t - t^3)$
$= \frac{\partial}{\partial r}(6rt) + \frac{\partial}{\partial t}(3r^2 - 3t^2)$
$= 6t + (-6t) = 0$

- $u(r,t) = \sin(r)\cosh(t)$.

Solution:
$u_{rr} + u_{tt} = \frac{\partial^2}{\partial r^2}(\sin(r)\cosh(t)) + \frac{\partial^2}{\partial t^2}(\sin(r)\cosh(t))$
$= \frac{\partial}{\partial r}(\cos(r)\cosh(t)) + \frac{\partial}{\partial t}(\sin(r)\sinh(t))$
$= -\sin(r)\cosh(t) + \sin(r)\cosh(t) = 0$

- $u(r,t) = \log(r^2 + t^2)$ for $r^2 + t^2 \neq 0$

Solution:
$u_{rr} + u_{tt} = \frac{\partial^2}{\partial r^2}(\log(r^2 + t^2)) + \frac{\partial^2}{\partial t^2}(\log(r^2 + t^2))$
$= \frac{\partial}{\partial r}\left(\frac{2r}{r^2+t^2}\right) + \frac{\partial}{\partial t}\left(\frac{2t}{r^2+t^2}\right)$
$= \frac{2t^2-2r^2}{(r^2+t^2)^2} + \frac{2r^2-2t^2}{(r^2+t^2)^2} = 0.$
The condition $r^2 + t^2 \neq 0$ is needed so that the expression $\log(r^2 + t^2)$ makes sense.

- $u(r,t) = e^r\cos(t) + 3r^2t - t^3 + \sin(r)\cosh(t)$

Solution:
Notice that the PDE is linear. Since the function in part (v) is the sum of the functions in the first three parts, by the superposition principle, it satisfies the PDE.

2. Verify that the each of the following functions satisfy the PDE "$u_{\theta\theta} - c^2 u_{rr} = 0$", where c is a real constant.

   - $u(r, \theta) = \sin(r - c\theta)$

Solution:
$u_{\theta\theta} - c^2 u_{rr} = \frac{\partial^2}{\partial\theta^2}(\sin(r - c\theta)) - c^2 \frac{\partial^2}{\partial r^2}(\sin(r - c\theta))$
$= \frac{\partial}{\partial\theta}(-c\cos(r - c\theta)) - c^2 \frac{\partial}{\partial r}(\cos(r - c\theta))$
$= -c^2 \sin(r - c\theta) - c^2(-\sin(r - c\theta)) = 0$

   - $u(r, \theta) = \log(r + c\theta)$ for $r + c\theta > 0$

Solution:
$u_{\theta\theta} - c^2 u_{rr} = \frac{\partial^2}{\partial\theta^2}(\log(r + c\theta)) - c^2 \frac{\partial^2}{\partial r^2}(\log(r + c\theta))$
$= \frac{\partial}{\partial\theta}\left(\frac{c}{r+c\theta}\right) - c^2 \frac{\partial}{\partial r}\left(\frac{1}{r+c\theta}\right)$
$= \frac{-c^2}{(r+c\theta)^2} - c^2\left(-\frac{1}{(r+c\theta)^2}\right) = 0$

   - $u(r, \theta) = \cos(ar)\sin(ca\theta)$ for any real constant a

Solution:
$u_{\theta\theta} - c^2 u_{rr} = \frac{\partial^2}{\partial\theta^2}(\cos(ar)\sin(ca\theta)) - c^2 \frac{\partial^2}{\partial r^2}(\cos(ar)\sin(ca\theta))$
$= \frac{\partial}{\partial\theta}(ca\cos(ar)\cos(ca\theta)) - c^2 \frac{\partial}{\partial r}(-a\sin(ar)\sin(ca\theta))$
$= -(ca)^2 \cos(ar)\sin(ca\theta) - c^2(-a^2\cos(ar)\sin(ca\theta)) = 0$

   - $u(r, \theta) = e^{r+c\theta} + e^{r-c\theta}$

Solution:
$u_{\theta\theta} - c^2 u_{rr} = \frac{\partial^2}{\partial\theta^2}(e^{r+c\theta} + e^{r-c\theta}) - c^2 \frac{\partial^2}{\partial r^2}(e^{r+c\theta} + e^{r-c\theta})$
$= \frac{\partial}{\partial\theta}(ce^{r+c\theta} - ce^{r-c\theta}) - c^2 \frac{\partial}{\partial r}(e^{r+c\theta} + e^{r-c\theta})$
$= (c^2 e^{r+c\theta} + c^2 e^{r-c\theta}) - c^2\left(e^{r+c\theta} + e^{r-c\theta}\right) = 0$

3. Verify that the each of the following functions satisfy the PDE "$u_t - ku_{xx} = 0$", where k is a real constant.

- $u(x,t) = x^2 + 2kt$

Solution:
$u_t - ku_{xx} = \frac{\partial}{\partial t}(x^2 + 2kt) - k\frac{\partial^2}{\partial x^2}(x^2 + 2kt)$
$= 2k - k\frac{\partial}{\partial x}(2x)$
$= 2k - 2k = 0$

- $u(x,t) = e^{-kt}\sin(x)$

Solution:
$u_t - ku_{xx} = \frac{\partial}{\partial t}(e^{-kt}\sin(x)) - k\frac{\partial^2}{\partial x^2}(e^{-kt}\sin(x))$
$= -ke^{-kt}\sin(x) - k\frac{\partial}{\partial x}(e^{-kt}\cos(x))$
$= -ke^{-kt}\sin(x) - k(-e^{-kt}\sin(x)) = 0$

- $u(x,t) = e^{kt}\cosh(x)$

Solution:
$u_t - ku_{xx} = \frac{\partial}{\partial t}(e^{kt}\cosh(x)) - k\frac{\partial^2}{\partial x^2}(e^{kt}\cosh(x))$
$= ke^{kt}\cosh(x) - k\frac{\partial}{\partial x}(e^{kt}\sinh(x))$
$= ke^{kt}\cosh(x) - ke^{kt}\cosh(x) = 0$

- $u(x,t) = e^{-a^2kt}\cos(ax)$ for any real constant a

Solution:
$u_t - ku_{xx} = \frac{\partial}{\partial t}(e^{-a^2kt}\cos(ax)) - k\frac{\partial^2}{\partial x^2}(e^{-a^2kt}\cos(ax))$
$= -a^2ke^{-a^2kt}\cos(ax) - k\frac{\partial}{\partial x}(-ae^{-a^2kt}\sin(ax))$
$= -a^2ke^{-a^2kt}\cos(ax) - k(-a^2e^{-a^2kt}\cos(ax)) = 0$

4. Classify each of the following 2nd order PDEs according to their homogeneity (homogeneous or nonhomogeneous), linearity (linear or non linear), coefficients (constant or non constant) and, when the PDE is linear, type (parabolic, hyperbolic or elliptic).

   Note: An equation may have different types in different regions of the domain of the function.

   (a) $(1+y^2)u_{xx} + e^{-\frac{x^2}{2}}u_{yy} - xu_x + (3-x^2)u_y = 0$ [Past exam question]

Solution:
The PDE is homogeneous, linear (since none of the coefficients is a function of u) and has non constant coefficients (because the coefficients are functions

that depend on x and y). Moreover, since $0^2 - 4(1+y^2)e^{-\frac{x^2}{2}} < 0$ for every possible values of (x, y), we conclude that the PDE is elliptic.

(b) $u_{xx} + 4xy + e^y u_y = (x+y)^2 u_x$ [Past exam question]

Solution:

The PDE is not homogeneous (because of the term $4xy$), linear and has non constant coefficients. Moreover, since $0 - 4(0 \cdot 1) = 0$, we conclude that the PDE is parabolic.

(c) $u_{r\theta} - 6u_{rr} + uu_{\theta\theta} - e^r u_r = 4x$

Solution:
The PDE is not homogeneous (because of the term $4x$), not linear (because of the term $uu_{\theta\theta}$) and has non constant coefficients.

(d) $6u_{tt} + 12u_{rr} - 24u_r + 20u_t = 42$

Solution:
The PDE is not homogeneous (because of the non-zero term 42), linear and has constant coefficients. Moreover, since $0 - 4(6 \cdot 12) < 0$ for every possible values of (r, t), we conclude that the PDE is elliptic.

(e) $u_{xx} + \sin(x)u_x + e^{\pi y}u_y - 10xyu_{yy} = 0$

Solution:
The PDE is homogeneous, linear and has not constant coefficients. Moreover, since $0 - 4(1 \cdot (-10xy)) = 40xy$ we conclude that the PDE is hyperbolic for $xy > 0$, parabolic for $xy = 0$ and elliptic for $xy < 0$.

5. **(Calculus review)** Compute the following integrals.

(a) $\int_0^\pi \cos(nx)\,dx$, for n a positive integer

Solution:
$\int_0^\pi \cos(nx)\,dx = \frac{1}{n}\sin(nx)\big|_0^\pi = \frac{1}{n}\sin(n\pi) - \frac{1}{n}\sin(0) = 0 - 0 = 0$

(b) $\int_{-\pi}^\pi x\sin(nx)\,dx$, for n a positive integer. [Past exam question]

Solution:
Note first of all that, since x and $\sin(nx)$ are odd functions, the function $x\sin(nx)$ is even. Thus, we have:
$\int_{-\pi}^\pi x\sin(nx)\,dx = 2\int_0^\pi x\sin(nx)\,dx$
Integrating by parts and using the result in a) we get:
$\int_0^\pi x\sin(nx)\,dx = -\frac{x\cos(nx)}{n}\Big|_0^\pi + \int_0^\pi \frac{\cos(nx)}{n}\,dx = -\frac{\pi\cos(n\pi)}{n} + 0 = \frac{\pi(-1)^{n+1}}{n}$

Thus we conclude that $\int_{-\pi}^\pi x\sin(nx)\,dx = \frac{2\pi(-1)^{n+1}}{n}$.

(c) $\int_0^{2\pi} \cos(2x)\cos(x)\,dx$

Solution:
Integrating by parts twice, we get:
$\int_0^{2\pi} \cos(2x)\cos(x)\,dx = \cos(2x)\sin(x)\big|_0^{2\pi} - \int_0^{2\pi} -2\sin(2x)\sin(x)\,dx = -2\int_0^{2\pi}\sin(2x)\sin(x)\,dx = 2\sin(2x)\cos(x)\big|_0^{2\pi} + 2\int_0^{2\pi} -2\cos(2x)\cos(x)\,dx = -4\int_0^{2\pi}\cos(2x)\cos(x)\,dx$

Thus, we conclude that $\int_0^{2\pi}\cos(2x)\cos(x)\,dx = 0$.

(d) $\int_0^{2\pi} e^{-x}\sin(nx)\,dx$, for n an integer

Solution:
If $n = 0$ then the integral is 0 since $\sin(0) = 0$. Consider now $n \neq 0$. Integrating by parts twice, we get:
$\int_0^{2\pi} e^{-x} \sin(nx)\, dx = -\frac{e^{-x}\cos(nx)}{n}\Big|_0^{2\pi} - \int_0^{2\pi} \frac{e^{-x}\cos(nx)}{n}\, dx = \frac{1 - e^{-2\pi}}{n} - \frac{e^{-x}\sin(nx)}{n^2}\Big|_0^{2\pi} + \int_0^{2\pi} -\frac{e^{-x}\sin(nx)}{n^2}\, dx = \frac{1 - e^{-2\pi}}{n} - \frac{1}{n^2}\int_0^{2\pi} e^{-x}\sin(nx)\, dx$
And so we can conclude that $\int_0^{2\pi} e^{-x} \sin(nx)\, dx = \frac{(1 - e^{-2\pi})n}{n^2 + 1}$.

(e) $\int_0^{2\pi} \sin(3x)^2\, dx$

Solution:
Integrating by parts twice, we get: $\int_0^{2\pi} \sin(3x)^2\, dx = -\frac{\sin(3x)\cos(3x)}{3}\Big|_0^{2\pi} + \int_0^{2\pi} \cos(3x)^2\, dx = \int_0^{2\pi} \cos(3x)^2\, dx = \int_0^{2\pi} 1 - \sin(3x)^2\, dx = 2\pi - \int_0^{2\pi} \sin(3x)^2\, dx$.

Thus, we can conclude that $\int_0^{2\pi} \sin(3x)^2\, dx = \pi$.