Analysis III (BAUG) Prof. Dr. Alessandro Sisto

Exam

Question 1:

Consider the following IBVP for $u(x,t)$:

$$\begin{aligned} \text{PDE:} u_t &= 5u_{xx} && \text{in } \Omega = (0,\pi)\times(0,\infty);\\ \text{BC:} u_x(0,t) &= u_x(\pi,t) = 0 && \text{for all } t>0;\\ \text{IC:} u(x,0) &= 406\cos(5x)(\cos(5x)+1) && \text{for all } x\in(0,\pi). \end{aligned}$$

Suppose that the solution $u(x,t)$ has the form:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} e^{c_n t}\left(a_n \cos\left(\frac{n\pi}{\pi}x\right) + b_n \sin\left(\frac{n\pi}{\pi}x\right)\right).$$

Then

1. Compute a_0; **203**
2. Compute a_5; **406**
3. Compute b_5; **0**
4. Compute c_5; **-125**
5. Compute a_{10}. **203**

Solution:
From $\cos^2(\theta) = \frac{1}{2}(1+cos(2\theta))$ we get

$$u(x,t) = 203 + 406e^{-5\cdot 5^2 t^2}\cos(5x) + 203e^{5\cdot 10^2 t^2}\cos(10x).$$

Thus, $a_0 = a_{10} = 203, a_5 = 406 b_5 = 0, c_5 = -125$.

Question 2:

Consider the following IBVP for $u(x,t)$:

$$\text{PDE: } u_{tt} = 9u_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0;$$

$$\text{IC: } u(x,0) = \begin{cases} x+4 & \text{for } x \in [-4,-2] \\ -x & \text{for } x \in [-2,2] \\ x-4 & \text{for } x \in [2,4] \\ 0 & \text{otherwise} \end{cases}$$

$$u_t(x,0) = \begin{cases} -6\pi\cos\left(\frac{\pi}{2}x\right) & \text{for } x \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$

Compute

1. Compute $u(12\pi, 1)$ **0**
2. Compute $u\left(0, \frac{1}{3}\right)$ **-4**
3. Compute $2u\left(\frac{1}{2}, \frac{5}{6}\right)$ **-7**
4. Compute $\lim_{t\to\infty} u(e^{\pi}, t)$ **-4**
5. is there (x,t) such that $u(x,t) \geq \pi$? **no**

Solution:
Using d'Alambert formula we get:

$$u(12\pi, 1) = \frac{f(12\pi+3) + f(12\pi-3)}{2} + \frac{1}{6}\int_{12\pi-3}^{12\pi+3} g(x)dx = 0,$$

since both $f(x)$ and $g(x)$ are zero for $x \geq 4$.

Now, we compute the following:

$$\frac{1}{6}\int_{-1}^{1} -6\pi\cos\left(\frac{\pi}{2}x\right)dx = -\pi\left[\frac{2}{\pi}\sin\left(\frac{\pi}{2}x\right)\right]_{-1}^{1} = -2(1+1) = -4$$

Thus, we have:

$$u\left(0,\frac{1}{3}\right) = \frac{f(1)+f(-1)}{2} + \frac{1}{6}\int_{-1}^{1} g(x)dx = -4$$
$$2u\left(\frac{1}{2},\frac{5}{6}\right) = 2\left(\frac{f(3)+f(-2)}{2} + \int_{-2}^{3} g(x)dx\right) = 2\left(\frac{-1+2}{2} - 4\right) = -7$$
$$\lim_{t\to\infty} u(e^{\pi},t) = \frac{1}{6}\int_{-1}^{1} -6\pi\cos\left(\frac{\pi}{2}x\right) dx = -4$$

For the last question, note that $g(x)$ is always negative, and the maximum value reached by $f(x)$ is 2. Thus, the maximum of the value of the solution is 2.

Question 3:

Suppose we have a Beam of length 12 (parametrized by $0 \le x \le 12$) embedded in the wall at the end $x = 0$ and free at the other. Suppose $EI = 3$ and suppose that we apply a force $F = 3$ downwards at the point $x = 4$.

Let $y(x)$ be the deflection curve.

Compute:

1. $y''(12)$ **0**
2. $y'(12)$ **-8**
3. $y'''(0)$ **1**
4. $3y(2)$ **-20**
5. $y'(8)$ **-8**

Solution:
The ODE we want to solve is

$$3y''''(x) = -3\delta(x-4)$$

with initial conditions $y(0) = y'(0) = y''(12) = y'''(12) = 0$. Taking the Laplace transform we obtain:

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = -e^{-4s};$$
$$s^4Y(s) - sy''(0) - y'''(0) = -e^{-4s}.$$

Let $A = y''(0)$ and $B = y'''(0)$. We have

$$s^4 Y(s) - As - B = -e^{-4s};$$
$$Y(s) = -\frac{e^{-4s}}{s^4} + \frac{A}{s^3} + \frac{B}{s^4}.$$

Taking the Laplace inverse we obtain:

$$y(x) = -\frac{1}{6}u(x-4)(x-4)^3 + \frac{A}{2}x^2 + \frac{B}{6}x^3$$

Using the fact that $u(x-4) = 1$ around $x = 12$ we obtain $y''(12) = -8 + A + 12B$ and $y'''(12) = -1 + B$. Hence we have $B = 1$ and $A = -4$. Thus,

$$y(x) = -\frac{1}{6}u(x-4)(x-4)^3 - 2x^2 + \frac{1}{6}x^3.$$

Question 4:

Solve the following ODE:

$$y''(t) + y(t) = t,$$

assuming $y(0) = 0$, $y'(0) = 2$.

1. Compute $y''(0)$ **0**
2. Compute $y(\frac{\pi}{2}) - \frac{\pi}{2}$ **1**
3. Compute $y'\left(\frac{3}{2}\pi\right)$ **1**
4. Compute y contains a term of the form cx^2, for some non-zero coefficient c **No**
5. if $t > 3$, then $y(t) > 2$. **yes**

Solution:
Using the Laplace transform we obtain:

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + Y(s) &= \frac{1}{s^2}; \\ s^2Y(s) - 2 + Y(s) &= \frac{1}{s^2}; \\ Y(s)(s^2+1) &= \frac{1}{s^2} + 2; \\ Y(s) &= \frac{1}{s^2(s^2+1)} + \frac{2}{s^2+1}. \end{aligned}$$

Using partial fractions we have:

$$\begin{aligned} \frac{1}{s^2(s^2+1)} &= \frac{A}{s^2} + \frac{B}{s^2+1}; \\ &= \frac{A(s^2+1) + s^2B}{s^2(s^2+1)}; \end{aligned}$$

And thus $A = 1, B = -1$. Hence,

$$Y(s) = \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{2}{s^2+1}.$$

Thus,

$$y(t) = t + \sin(t).$$

Question 5:

Let $f\colon [-2,2] \to \mathbb{R}$ be the following function:

$$f(x) = x + |x|.$$

Compute the Fourier series of f.

Solution:
There are two ways to solve this exercise. The first one is to separate $f(x)$ in x and

$|x|$, observe that one is odd and the other is even, and use the symmetries to compute coefficients for both of them. Alternatively, one may observe that $f(x)$ is constantly zero on $[-2,0]$ and equal to $2x$ on $[0,2]$. Then we have:

$$a_0 = \frac{1}{4}\int_0^2 2x dx = \frac{1}{4}\left[x^2\right]_0^2 = 1$$

$$\begin{aligned} a_n &= \frac{1}{2}\int_0^2 2x\cos\left(\frac{n\pi}{2}x\right)dx = \frac{2}{n\pi}\left[x\sin\left(\frac{n\pi}{2}x\right)\right]_0^2 - \frac{2}{n\pi}\int_0^2 \sin\left(\frac{n\pi}{2}x\right)dx = \\ &= 0 + \frac{4}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}x\right)\right]_0^2 = \frac{4}{n^2\pi^2}\left((-1)^n - 1\right). \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{2}\int_0^2 2x\sin\left(\frac{n\pi}{2}x\right)dx = \frac{-2}{n\pi}\left[x\cos\left(\frac{n\pi}{2}x\right)\right]_0^2 + \frac{2}{n\pi}\int_0^2 \cos\left(\frac{n\pi}{2}x\right)dx = \\ &= \frac{4}{n\pi}(-1)^{n+1} + \frac{4}{n^2\pi^2}\left[\sin\left(\frac{n\pi}{2}x\right)\right]_0^2 = \frac{4}{n\pi}(-1)^{n+1} \end{aligned}$$

Question 6:

Consider the following IBVP.

$$\begin{aligned} &\text{PDE:} && u_{tt} = u_{xx} && \text{for } t>0, x\in(0,L) \\ &\text{BC:} && u(0,t) = u(L,t) = 0 && \text{for } t>0 \\ &\text{IC:} && u(x,0) = f(x) && \text{for } x\in[0,L] \\ & && u_t(x,0) = 0 && \text{for } x\in[0,L]. \end{aligned}$$

Let $u(x,t)$ be a solution for the above IBVP and let $h\colon [0,L]\to\mathbb{R}$ be defined as $h(s) = u(s,s)$. Compute $\int_0^L h(s)ds$.

Solution:
Using the formula we known that $u(x,t)$ has the form:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(b_n\cos\left(\frac{n\pi}{L}t\right) + \frac{c_n L}{n\pi}\sin\left(\frac{n\pi}{L}t\right)\right),$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$
$$c_n = \frac{2}{L} \int_0^L u_t(x,0) \sin\left(\frac{n\pi}{L} x\right) dx = 0.$$

Thus, integrating $u(s,s)$ we obtain:

$$\int_0^L h(s)ds = \int_0^L u(s,s)ds = \int_0^L \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} s\right) \cos\left(\frac{n\pi}{L} s\right) ds.$$

By orthogonality of the trigonometric polynomials, the above is always zero.