

Midterm

Question 1:

For each of the following PDEs, decide if it is elliptic, parabolic, hyperbolic or of mixed type.

- (a) $e^{4t}u_{xx} + 2e^t u_{xt} - u_x = \sin(x)u - u_{tt}$
- (b) $5u_x + 32u_{yy} - 4u_y = 12u_{xx}$
- (c) $(\frac{47}{3} + \sin(x))u_{xy} - \cos(x)u_{xx} = u_x - (e^{4t} + 2)u_y$
- (d) $u_{xx} + u_{yy} = \frac{1}{5+x^4}u_x + 2\cos(3\pi)u_{xy}$

Question 2:

Consider the following IBVP.

$$\begin{aligned} \text{PDE:} \quad & u_t = 7u_{xx} && \text{for } t > 0, x \in (0, L) \\ \text{BC:} \quad & u(0, t) = u(L, t) = 0 && \text{for } t > 0 \\ \text{IC:} \quad & u(x, 0) = \sin\left(\frac{15\pi}{3L}x\right) \cos\left(\frac{3\pi}{L}x\right) - \frac{1}{7} \sin\left(\frac{2\pi}{L}x\right) && \text{for } x \in [0, L]. \end{aligned}$$

- (a) Which of the following problems does the IBVP model?
 - (i) ► The heat flow on a laterally insulated rod with the temperature fixed at both ends.
 - (ii) The heat flow on a completely insulated rod.
 - (iii) The heat flow on a circular wire.
 - (iv) The vibration of a finite string.
 - (v) The vibration of an infinite string.
- (b) Suppose that the solution $u(x, t)$ has the form:

$$u(x, t) = a_0 + \sum_{n=1}^{\infty} e^{c_n t} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right).$$

Decide which of the following are true and which are false.

- (i) $a_2 = \frac{5}{14}$ (ii) $b_8 = \frac{3}{5}$ (iii) $b_6 = 0$ (iv) $c_2 = -14\left(\frac{2\pi}{L}\right)^2$ (v) $b_2 = \frac{5}{14}$

Question 3:

Consider the following IBVP.

$$\begin{array}{ll} \text{PDE:} & u_t = 9u_{xx} & \text{for } t > 0, x \in (-5, 5) \\ \text{BC:} & u(-5, t) = u(5, t) & \text{for } t > 0 \\ & u_x(-5, t) = u_x(5, t) & \text{for } t > 0 \\ \text{IC:} & u(x, 0) = \begin{cases} 1 - \cos\left(\frac{2\pi}{5}x\right) & \text{for } x \in [-5, 0] \\ 0 & \text{for } x \in [0, 5] \end{cases} \end{array}$$

- (a) Which of the following problems does the IBVP model?
- (i) The heat flow on a laterally insulated rod with the temperature fixed at both ends.
 - (ii) The heat flow on a completely insulated rod.
 - (iii) ► The heat flow on a circular wire.
 - (iv) The vibration of a finite string.
 - (v) The vibration of an infinite string.
- (b) Suppose that the solution $u(x, t)$ has the form:

$$u(x, t) = a_0 + \sum_{n=1}^{\infty} e^{c_n t} \left(a_n \cos\left(\frac{n\pi}{5}x\right) + b_n \sin\left(\frac{n\pi}{5}x\right) \right).$$

Decide which of the following are true and which are false.

(i) $b_3 = \frac{8}{15\pi}$ (ii) $b_1 = 0$ (iii) $b_2 = 4\pi$ (iv) $c_2 = -9\left(\frac{2\pi}{5}\right)^2$

Question 4:

Consider the following IVP.

$$\begin{array}{ll} \text{PDE:} & u_{tt} = u_{xx} & \text{for } t > 0, -\infty < x < \infty \\ \text{IC:} & u(x, 0) = \frac{1}{x^2 + 1} \cos(\pi x) & \text{for } -\infty < x < \infty \\ & u_t(x, 0) = \begin{cases} 1 & \text{for } x \in [n, n+1], n \text{ even,} \\ 0 & \text{else.} \end{cases} \end{array}$$

- (a) Which of the following problems does the IVP model?

- (i) The heat flow on a laterally insulated rod with the temperature fixed at both ends.
 - (ii) The heat flow on a completely insulated rod.
 - (iii) The heat flow on a circular wire.
 - (iv) The vibration of a finite string.
 - (v) ► The vibration of an infinite string.
- (b) Let $u(x, t)$ be a solution. Decide which of the following are true and which are false.
- (i) $u(0, 0) = 0$
 - (ii) $u(1, 0) = -\frac{1}{2}$
 - (iii) $u(3, \frac{1}{2}) = \frac{16}{17}$
 - (iv) $u(5, 2) = \frac{47}{50}$

Question 5:

Consider the following IBVP.

$$\begin{array}{ll}
 \text{PDE:} & u_{tt} = 9u_{xx} & \text{for } t > 0, x \in (0, \pi) \\
 \text{BC:} & u(0, t) = u(\pi, t) = 0 & \text{for } t > 0 \\
 \text{IC:} & u(x, 0) = f(x) & \text{for } x \in [0, \pi] \\
 & u_t(x, 0) = g(x) & \text{for } x \in [0, \pi].
 \end{array}$$

Let u_1 be the solution of the IBVP for $f(x) = \sin(x)$, $g(x) = \frac{9}{5} \cos(\log(\sin(\frac{2\pi}{L}x)))$, and let u_2 be the solution for $f(x) = \sin(x)$ and $g(x) = -\frac{9}{5} \cos(\log(\sin(\frac{2\pi}{L}x)))$. Decide which of the following are true and which are false.

- (i) $u_1 = -u_2$
- (ii) $(u_1 + u_2)(1, 1) = \frac{1}{2}$
- (iii) $(u_1 + u_2)(2, \frac{2}{3}) = \sin(4)$
- (iv) $(u_1 + u_2)(\pi, 2\pi) = 0$

Question 6:

Consider the following IBVP.

$$\begin{array}{ll}
 \text{PDE:} & u_t = u_{xx} & \text{for } t > 0, x \in (0, \pi) \\
 \text{BC:} & u(0, t) = u_x(\pi, t) = 0 & \text{for } t > 0 \\
 \text{IC:} & u(x, 0) = \frac{1}{5} \left(\sin(2x) \cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right) \cos(2x) \right) & \text{for } x \in [0, \pi].
 \end{array}$$

Note that in the BC we have $\frac{\partial u}{\partial x}(\pi, t) = 0$, and not $u(\pi, t) = 0$. Decide which of the following are true and which are false.

- (i) $u(1, \frac{5}{2}) = \frac{1}{5} e^{-\frac{25}{4}} \cos(\frac{5}{2})$
- (ii) $u(\frac{\pi}{5}, 4) = \frac{e^{-25}}{5}$
- (iii) $u(\frac{2\pi}{5}, 6) = 0$