**Analysis III (BAUG)** Prof. Dr. Alessandro Sisto **Version A**

# Midterm

## Question 1:

For each of the following PDEs, decide if it is elliptic, parabolic, hyperbolic or of mixed type.

(a) $e^{4t}u_{xx} + 2e^t u_{xt} - u_x = \sin(x)u - u_{tt}$

(b) $5u_x + 32u_{yy} - 4u_y = 12u_{xx}$

(c) $(\frac{47}{3} + \sin(x))u_{xy} - \cos(x)u_{xx} = u_x - (e^{4t} + 2)\, u_y$

(d) $u_{xx} + u_{yy} = \frac{1}{5+x^4}u_x + 2\cos(3\pi)u_{xy}$

**Solution:**
We recall that for an equation of the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$$

we need to compute the sign of $B^2 - 4AC$.

In the first equation we have: $(2e^t)^2 - 4e^{4t} = 4e^{2t}(1 - e^{2t})$. In particular, it is hyperbolic when $t < 0$, elliptic when $t > 0$ and parabolic when $t = 0$.

We realized that it is not clear whether $t$ ranges from $-\infty$ to $\infty$, or if $t > 0$. Indeed, the first option can be assumed since there are no specifications on the range, and the second since we only considered PDEs with $t > 0$. For this reason, we accept both "mixed" and "elliptic" as correct solutions.

In the second equation we have $0 - 4(-12)32 > 0$. Thus is hyperbolic.

In the third equation we have $\left(\frac{47}{3} + \sin(x)\right)^2 - 0 > 0$. Thus is hyperbolic.

In the last equation we have $(-2(-1))^2 - 4(1 \cdot 1) = 0$. Thus is parabolic.

## Question 2:

Consider the following IBVP.

| | | |
|---|---|---|
| PDE: | $u_t = 7u_{xx}$ | for $t > 0, x \in (0, L)$ |
| BC: | $u(0,t) = u(L,t) = 0$ | for $t > 0$ |
| IC: | $u(x,0) = \sin\left(\frac{15\pi}{3L}x\right)\cos\left(\frac{3\pi}{L}x\right) - \frac{1}{7}\sin\left(\frac{2\pi}{L}x\right)$ | for $x \in [0, L]$. |

(a) Which of the following problems does the IBVP model?

(i) ► The heat flow on a laterally insulated rod with the temperature fixed at both ends.

(ii) The heat flow on a completely insulated rod.

(iii) The heat flow on a circular wire.

(iv) The vibration of a finite string.

(v) The vibration of an infinite string.

(b) Suppose that the solution $u(x,t)$ has the form:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} e^{c_n t}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right).$$

Decide which of the following are true and which are false.

(i) $a_2 = \frac{5}{14}$ (ii) $b_8 = \frac{3}{5}$ (iii) $b_6 = 0$ (iv) $c_2 = -14\left(\frac{2\pi}{L}\right)^2$ (v) $b_2 = \frac{5}{14}$

**Solution:**
We observe that $\sin\left(\frac{15\pi}{3L}x\right) = \sin\left(\frac{5\pi}{L}x\right)$.
From $\sin(\theta)\cos(\phi) = \frac{1}{2}(\sin(\theta+\phi) + \sin(\theta-\phi))$, we obtain

$$\phi(x) = \frac{1}{2}\left(\sin\left(\frac{8\pi}{L}x\right) + \sin\left(\frac{2\pi}{L}x\right)\right) - \frac{1}{7}\sin\left(\frac{2\pi}{L}x\right).$$

Thus $\phi(x) = \frac{5}{14}\sin\left(\frac{2\pi}{L}x\right) + \frac{1}{2}\sin\left(\frac{8\pi}{L}x\right)$. Thus, $b_2 = \frac{5}{14}$, $b_8 = \frac{1}{2}$. All the $a_n$ are zero, and $c_n = -7\left(\frac{n\pi}{L}\right)^2$.

# Question 3:

Consider the following IBVP.

| | | |
|---|---|---|
| PDE: | $u_t = 9u_{xx}$ | for $t > 0, x \in (-5,5)$ |
| BC: | $u(-5,t) = u(5,t)$ | for $t > 0$ |
| | $u_x(-5,t) = u_x(5,t)$ | for $t > 0$ |
| IC: | $u(x,0) = \begin{cases} 1 - \cos\left(\frac{2\pi}{5}x\right) \text{ for } x \in [-5,0] \\ 0 \text{ for } x \in [0,5] \end{cases}$ | |

(a) Which of the following problems does the IBVP model?

(i) The heat flow on a laterally insulated rod with the temperature fixed at both ends.

(ii) The heat flow on a completely insulated rod.

(iii) ▶ The heat flow on a circular wire.

(iv) The vibration of a finite string.

(v) The vibration of an infinite string.

(b) Suppose that the solution $u(x,t)$ has the form:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} e^{c_n t}\left(a_n \cos\left(\frac{n\pi}{5}x\right) + b_n \sin\left(\frac{n\pi}{5}x\right)\right).$$

Decide which of the following are true and which are false.

(i) $b_3 = \frac{8}{15\pi}$ (ii) $b_1 = 0$ (iii) $b_2 = 4\pi$ (iv) $c_2 = -9\left(\frac{2\pi}{5}\right)^2$

**Solution:**
We have:

$$\begin{aligned}
b_n =& \frac{1}{5}\int_{-5}^{5} \phi(x)\sin\left(\frac{n\pi}{5}x\right)dx = \\
=& \frac{1}{5}\left(\int_{-5}^{0} \sin\left(\frac{n\pi}{5}x\right)dx - \int_{-5}^{0}\cos\left(\frac{2\pi}{5}x\right)\sin\left(\frac{n\pi}{5}x\right)dx\right) = \\
=& \frac{1}{5}\left(\frac{-5}{n\pi}\cos\left(\frac{n\pi}{5}x\right)\Big|_{-5}^{0}\right) - \frac{1}{10}\left(\int_{-5}^{0}\sin\left(\frac{(n+2)\pi}{5}x\right)dx + \int_{-5}^{0}\sin\left(\frac{(n-2)\pi}{5}x\right)dx\right) = \\
=& \begin{cases} 0 \text{ if } n \text{ is even} \\ -\frac{2}{n\pi} + \frac{1}{(n+2)\pi} + \frac{1}{(n-2)\pi} = \frac{8}{n(n-2)(n+2)\pi} \text{ otherwise} \end{cases}
\end{aligned}$$

Thus we have $b_1 = -\frac{8}{3\pi}$, $b_2 = 0$, $b_3 = \frac{8}{15\pi}$. By $c_n = -\alpha^2\left(\frac{n\pi}{L}\right)^2$, we get $c_2 = -9\left(\frac{2\pi}{5}\right)^2$.

## Question 4:

Consider the following IVP.

PDE: $u_{tt} = u_{xx}$ for $t > 0, -\infty < x\infty$

IC: $u(x,0) = \frac{1}{x^2+1}\cos(\pi x)$ for $-\infty < x < \infty$

$$u_t(x,0) = \begin{cases} 1 \text{ for } x \in [n, n+1], n \text{ even}, \\ 0 \text{ else.} \end{cases}$$

(a) Which of the following problems does the IVP model?

- (i) The heat flow on a laterally insulated rod with the temperature fixed at both ends.
- (ii) The heat flow on a completely insulated rod.
- (iii) The heat flow on a circular wire.
- (iv) The vibration of a finite string.
- (v) ► The vibration of an infinite string.

(b) Let $u(x,t)$ be a solution. Decide which of the following are true and which are false.

(i) $u(0,0) = 0$ (ii) $u(1,0) = -\frac{1}{2}$ (iii) $u(3, \frac{1}{2}) = \frac{16}{17}$ (iv) $u(5,2) = \frac{47}{50}$

**Solution:**

Using d'Alambert formula we have

$$u(x,t) = \frac{1}{2}\left(\frac{\cos((x+t)\pi)}{(x+t)^2+1} + \frac{\cos((x-t)\pi)}{(x-t)^2+1}\right) + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds,$$

where $g(s) = \begin{cases} 1 \text{ for } s \in [n, n+1], n \text{ even}, \\ 0 \text{ else.} \end{cases}$

Thus we get:

$$u(0,0) = \frac{1}{2}\left(\frac{1}{1} + \frac{1}{1}\right) + 0 = 1$$

$$u(1,0) = \frac{1}{2}\left(\frac{-1}{2} + \frac{-1}{2}\right) + 0 = -\frac{1}{2}$$

$$u(3, \frac{1}{2}) = \frac{1}{2}(0+0) + \frac{1}{2}\int_{2+\frac{1}{2}}^{3} ds = \frac{1}{4}$$

$$u(5,2) = \frac{1}{2}\left(\frac{-1}{49+1} + \frac{-1}{9+1}\right) + \frac{1}{2}\left(\int_4^5 ds + \int_6^7 ds\right) = \frac{1}{2}\left(\frac{-6}{50}\right) + \frac{1}{2}(2) = \frac{50-3}{50} = \frac{47}{50}$$

## Question 5:

Consider the following IBVP.

$$\begin{aligned}
\text{PDE:} &\quad u_{tt} = 9u_{xx} && \text{for } t > 0, x \in (0, \pi) \\
\text{BC:} &\quad u(0,t) = u(\pi,t) = 0 && \text{for } t > 0 \\
\text{IC:} &\quad u(x,0) = f(x) && \text{for } x \in [0, \pi] \\
&\quad u_t(x,0) = g(x) && \text{for } x \in [0, \pi].
\end{aligned}$$

Let $u_1$ be the solution of the IBVP for $f(x) = \sin(x)$, $g(x) = \frac{9}{5}\cos\left(\log\left(\sin\left(\frac{2\pi}{L}x\right)\right)\right)$, and let $u_2$ be the solution for $f(x) = \sin(x)$ and $g(x) = -\frac{9}{5}\cos\left(\log\left(\sin\left(\frac{2\pi}{L}x\right)\right)\right)$. Decide which of the following are true and which are false.

(i) $u_1 = -u_2$ (ii) $(u_1 + u_2)(1,1) = \frac{1}{2}$

(iii) $(u_1 + u_2)(2, \frac{2}{3}) = \sin(4)$ (iv) $(u_1 + u_2)(\pi, 2\pi) = 0$

**Solution:**

We observe that point (i) has to be false, since $u_1(x,0) = u_2(x,0)$ and there is a value $x$ such that $u_1(x,0) \neq 0$. The points (ii), (iii) and (iv) only involve $u_1 + u_2$. By the superposition principle, $u_1 + u_2$ satisfies the following IBVP:

$$\begin{aligned}
\text{PDE:} &\quad u_{tt} = 9u_{xx} && \text{for } t > 0, x \in (0, \pi) \\
\text{BC:} &\quad u(0,t) = u(\pi,t) = 0 && \text{for } t > 0 \\
\text{IC:} &\quad u(x,0) = 2\sin(x) && \text{for } x \in [0, \pi] \\
&\quad u_t(x,0) = 0 && \text{for } x \in [0, \pi].
\end{aligned}$$

Thus, $(u_1 + u_2)(x,t) = 2\sin(x)\cos(3t)$. Then we have:

$$\begin{aligned}
(u_1 + u_2)(1,1) &= 2\sin(1)\cos(3) \\
(u_1 + u_2)(2, \frac{2}{3}) &= 2\sin(2)\cos(2) = \sin(4) \\
(u_1 + u_2)(\pi, 2\pi) &= 2\sin(\pi)\cos(3\pi) = 0.
\end{aligned}$$

# Question 6:

Consider the following IBVP.

| | | |
|---|---|---|
| PDE: | $u_t = u_{xx}$ | for $t > 0, x \in (0, \pi)$ |
| BC: | $u(0,t) = u_x(\pi,t) = 0$ | for $t > 0$ |
| IC: | $u(x,0) = \dfrac{1}{5}\left(\sin(2x)\cos\left(\dfrac{x}{2}\right) + \sin\left(\dfrac{x}{2}\right)\cos(2x)\right)$ | for $x \in [0, \pi]$. |

Note that in the BC we have $\frac{\partial u}{\partial x}(\pi, t) = 0$, and not $u(\pi, t) = 0$. Decide which of the following are true and which are false.

(i)$u(1, \frac{5}{2}) = \frac{1}{5}e^{-\frac{25}{4}}\cos\left(\frac{5}{2}\right)$ (ii) $u(\frac{\pi}{5}, 4) = \frac{e^{-25}}{5}$ (iii) $u(\frac{2\pi}{5}, 6) = 0$

**Solution:**
Using the Ansatz $u(x,t) = X(x)T(t)$, we proceed as in the lecture to obtain

$$u(x,t) = \begin{cases} Ax + B \\ e^{\lambda^2 t}(Ae^{\lambda x} + Be^{-\lambda x}) \\ e^{-\lambda^2 t}(A\sin(\lambda x) + B\cos(\lambda x)). \end{cases}$$

It is easily verified that matching the initial condition $u(0,t) = 0$ provides respectively

$$\begin{aligned} B &= 0 \\ A &= -B \\ B &= 0. \end{aligned}$$

Now consider the condition $u_x(0, \pi) = 0$. Computing the derivatives we obtain, respectively

$$\begin{aligned} A &= 0 \\ A = B &= 0 \\ A = 0 \text{ or } \cos(\lambda\pi) = 0 \leftrightarrow \lambda = \frac{2n+1}{2} &\text{ for some } n. \end{aligned}$$

Thus, the only non-zero solution that we obtain is

$$u(x,t) = Ae^{-\left(\frac{2n+1}{2}\right)^2 t}\sin\left(\frac{2n+1}{2}x\right).$$

We want now to match the solution we found with our initial condition. We recall the trigonometric identity $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$. Thus we get $u(x,0) = \frac{1}{5}\sin\left(2x + \frac{x}{2}\right) = \frac{1}{5}\sin\left(\frac{5}{2}x\right)$. Luckily, the initial condition is already of the same form of the solution obtained with the Ansatz $u(x,t) = X(x)T(t)$, choosing $A = \frac{1}{5}$ and $n = 2$. Thus we have that the solution of the IBVP is

$$u(x,t) = \frac{1}{5}e^{-\frac{25}{4}t}\sin\left(\frac{5}{2}x\right).$$

Hence:

$$\begin{aligned} u(1, \frac{5}{2}) &= \frac{1}{5}e^{-\frac{125}{8}}\sin\left(\frac{5}{2}\right) \\ u(\frac{\pi}{5}, 4) &= \frac{1}{5}e^{-25}\sin\left(\frac{\pi}{2}\right) = \frac{e^{-25}}{5} \\ u(\frac{2\pi}{5}, 6) &= \frac{1}{5}e^{-\frac{75}{2}}\sin(\pi) = 0. \end{aligned}$$