Midterm

Question 1:

For each of the following PDEs, decide if it is elliptic, parabolic, hyperbolic or neither of them (called mixed type).

(a) $u_{xx} - (2 + \sin(x^3))^2 \, u_y = -\left(2 + \sin\left(x^3\right)\right)^2 u_{yy}$

■ elliptic; ☐ parabolic; ☐ hyperbolic; ☐ mixed type.

(b) $e^{2y} u_{xy} + e^y (u_{xx} + u_{yy}) = 4e^{4t}$

☐ elliptic; ☐ parabolic; ☐ hyperbolic; ■ mixed type.

(c) $\sin^2(xy)u_y + \sin^3(x^2y^2)u_x + \arcsin(xy)u = \frac{1}{x^2+1}$

☐ elliptic; ■ parabolic; ☐ hyperbolic; ☐ mixed type.

(d) $\cos(x)u_{xx} + e^y u_x + e^x u_y = -7u_{xy}$

☐ elliptic; ☐ parabolic; ■ hyperbolic; ☐ mixed type.

Solution:
We recall that for an equation of the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$$

we need to compute the sign of $B^2 - 4AC$.

In the first equation we have: $0 - 4\left(1 \cdot (2 + \sin(x^3))^2\right) < 0$. Thus it is elliptic.

In the second equation we have: $e^{2t} - 4e^t \cdot e^t = e^{2t}\left(e^{2t} - 4\right)$. Since for some values of t the above is positive, for other negative and for others 0, it is of mixed type.

In the third equation we don't have the terms u_{xx}, u_{xy} and u_{yy}. Thus it is parabolic.

In the last equation we have: $49 - 4\cos(x) > 0$. Thus it is hyperbolic.

Question 2:

Consider the following IBVP.

PDE:	$u_t = 7u_{xx}$	for $t > 0, x \in (-L, L)$
BC:	$u(-L,t) = u(L,t)$	
	$u_x(-L,t) = u_x(L,t)$	for $t > 0$
IC:	$u(x,0) = 5\cos^2\left(\frac{6\pi}{L}x\right) + \frac{\sin\left(\frac{9\pi}{L}x\right) + \sin\left(\frac{7\pi}{L}x\right)}{\cos\left(\frac{\pi}{L}x\right)}$	for $x \in [-L, L]$.

(a) Which of the following problems does the IBVP model?

(i) The heat flow on a laterally insulated rod with the temperature fixed at both ends.

(ii) The heat flow on a completely insulated rod.

▶ The heat flow on an insulated circular wire.

(iii) The vibration of a finite string.

(iv) The vibration of an infinite string.

(b) Suppose that the solution $u(x,t)$ has the form:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} e^{c_n t}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right).$$

Decide which of the following are true and which are false.

(i) $b_{1024} = \frac{2^4}{\pi}$	☐ True	■ False	☐ I don't know
(ii) $c_8 = -49\left(\frac{8\pi}{L}\right)^2$	☐ True	■ False	☐ I don't know
(iii) $b_8 = 2$	■ True	☐ False	☐ I don't know
(iv) $a_0 = \frac{5}{2}$	■ True	☐ False	☐ I don't know
(v) $a_1 2 = \frac{5}{2}$	■ True	☐ False	☐ I don't know

Solution:
From $\cos(\theta)^2 = \frac{1}{2}(1+\cos(2\theta))$, we obtain that the first summand is equal to $\frac{5}{2} + \frac{5}{2}\cos\left(\frac{12\pi}{L}x\right)$.

From $\sin(\theta) + \sin(\phi) = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$ we obtain that the second summand is equal to $2\sin\left(\frac{8\pi}{L}x\right)$. Thus

$$\phi(x) = \frac{5}{2} + \frac{5}{2}\cos\left(\frac{12\pi}{L}x\right) + 2\sin\left(\frac{8\pi}{L}x\right).$$

In particular $a_0 = \frac{5}{2}$, $a_{12} = \frac{5}{2}$ and $b_8 = 2$.

Question 3:

Consider the following IBVP.

PDE:	$u_t = u_{xx}$	for $t > 0, x \in (0, 2\pi)$
BC:	$u_x(0,t) = u_x(2\pi, t) = 0$	for $t > 0$
IC:	$u(x,0) = \begin{cases} 0 & \text{for } x \in [0,\pi] \\ \sin(x-\pi) & \text{for } x \in [\pi, 2\pi] \end{cases}$	

(a) Which of the following problems does the IBVP model?

(i) The heat flow on a laterally insulated rod with the temperature fixed at both ends.

▶ The heat flow on a completely insulated rod.

(ii) The heat flow on an insulated circular wire.

(iii) The vibration of a finite string.

(iv) The vibration of an infinite string.

(b) Suppose that the solution $u(x,t)$ has the form:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} e^{c_n t} \left(a_n \cos\left(\frac{n\pi}{2\pi}x\right) + b_n \sin\left(\frac{n\pi}{2\pi}x\right) \right).$$

Decide which of the following are true and which are false.

(i) $a_2 = 0$	■ True	☐ False	☐ I don't know
(ii) $a_8 = \frac{-2}{15\pi}$	■ True	☐ False	☐ I don't know
(iii) $a_0 = \frac{1}{\pi}$	■ True	☐ False	☐ I don't know
(iv) $b_3 = \frac{9}{25\pi^2}$	☐ True	■ False	☐ I don't know

Solution:
Recall that $\sin(x - \pi) = -\sin(x)$ and that $\sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha+\beta) - \sin(\alpha-\beta))$.

We have that $b_n = 0$ for all n

We have:

$$
\begin{aligned}
a_0 =& \frac{1}{2\pi}\int_0^{2\pi} u(x,0)dx = \frac{1}{2\pi}\int_\pi^{2\pi} -\sin(x)dx = \\
=& \frac{1}{2\pi}\left(\cos(x)|_\pi^{2\pi}\right) = \frac{1+1}{2\pi} = \frac{1}{\pi}.
\end{aligned}
$$

$$
\begin{aligned}
a_n =& \frac{1}{\pi}\int_0^{2\pi} u(x,0)\cos\left(\frac{n\pi}{2\pi}x\right)dx = \\
=& \frac{1}{\pi}\left(\int_\pi^{2\pi} -\sin(x)\cos\left(\frac{n}{2}x\right)dx\right) = \\
=& \frac{-1}{2\pi}\left(\int_\pi^{2\pi}\sin\left(\left(1+\frac{n}{2}\right)x\right)dx + \int_\pi^{2\pi}\sin\left(\left(1-\frac{n}{2}\right)x\right)dx\right) =
\end{aligned}
$$

(Assume that $n \neq 2$)

$$
\begin{aligned}
=& \frac{1}{(2+n)\pi}\cos\left(\left(1+\frac{n}{2}\right)x\right)\Big|_\pi^{2\pi} + \frac{1}{(2-n)\pi}\cos\left(\left(1-\frac{n}{2}\right)x\right)\Big|_\pi^{2\pi} = \\
=& \frac{1}{(2+n)\pi}\left(\cos(((2+n)\pi) - \cos\left(\frac{2+n}{2}\pi\right)\right) + \frac{1}{(2-n)\pi}\left(\cos(((2-n)\pi) - \cos\left(\frac{2-n}{2}\pi\right)\right) = \\
=& \begin{cases} \frac{-4}{(4-n^2)\pi} & \text{if } n \text{ is odd;} \\ \frac{8}{(4-n^2)\pi} & \text{if } n \equiv 0(\text{mod}4); \\ 0 & \text{if } n \equiv 2(\text{mod}4). \end{cases}
\end{aligned}
$$

If $n = 2$:

$$
a_2 = \frac{-1}{2\pi}\int_\pi^{2\pi}\sin(2x)dx = 0.
$$

Question 4:

Consider the following IVP.

PDE:	$u_{tt} = u_{xx}$	for $t > 0, -\infty < x < \infty$
IC:	$u(x,0) = \dfrac{x}{x^2+1}$	for $-\infty < x < \infty$

$$u_t(x,0) = \begin{cases} x+1 & \text{for } x \in [-1,0] \\ -x+1 & \text{for } x \in [0,1] \\ 0 & \text{else.} \end{cases}$$

(a) Which of the following problems does the IVP model?

- (i) The heat flow on a laterally insulated rod with the temperature fixed at both ends.
- (ii) The heat flow on a completely insulated rod.
- (iii) The heat flow on a circular wire.
- (iv) The vibration of a finite string.
- ▶ The vibration of an infinite string.

(b) Let $u(x,t)$ be a solution. Decide which of the following are true and which are false.

(i) $u(3,2) = \frac{67}{260}$	☐ True	■ False	☐ I don't know
(ii) $u(0,2) = -\frac{1}{2}$	☐ True	■ False	☐ I don't know
(iii) $u(1,1) = 0$	☐ True	■ False	☐ I don't know
(iv) $u(0,4) = \frac{1}{2}$	■ True	☐ False	☐ I don't know

Solution:

$$u(3,2) = \frac{1}{2}\left(\frac{5}{26} + \frac{2}{5}\right) + 0 = \frac{77}{260}$$

$$u(0,2) = 0 + \frac{1}{2} = \frac{1}{2}$$

$$u(1,1) = \frac{1}{2}\left(0 + \frac{2}{5}\right) + \frac{1}{4} = \frac{9}{20}$$

$$u(0,4) = 0 + \frac{1}{2} = \frac{1}{2}$$

Question 5:

Consider the following IBVP.

PDE:	$u_{tt} = c^2 u_{xx}$	for $t > 0, x \in (0, 20\pi)$
BC:	$u(0,t) = u(20\pi, t) = 0$	for $t > 0$
IC:	$u(x,0) = \sin(x)$	for $x \in [0, 20\pi]$
	$u_t(x,0) = 0$	for $x \in [0, 20\pi]$.

Let $u(x,t)$ be a solution. Decide which of the following are true and which are false.

(i) $(u)(10\pi, t) = 0$ for all t	■ True	☐ False	☐ I don't know
(ii) $u(11\pi, t) = 0$ for all t	■ True	☐ False	☐ I don't know
(iii) For any time t, there always is a point x such that $u(t,x) \neq 0$	☐ True	■ False	☐ I don't know
(iv) $u\left(\frac{\sqrt{2}}{2}, \frac{1}{c}\right) = 2$	☐ True	■ False	☐ I don't know

Solution:
We get that $b_{20} = 1$ and all the other b_n and c_n are zero. Thus the solution is:

$$u(x,t) = \sin(x)\cos(ct)$$

Since $\sin(10\pi) = \sin(11\pi) = 0$, the first two are true. The third is false, for instance take $t = \frac{\pi}{2c}$. The last one is false because sin and cos are always smaller or equal 1. Thus so it is their product (and so can never be equal to 2).

Question 6:

Consider the following IBVP.

PDE:	$u_t = 4u_{xx}$	for $t > 0, x \in (0, \pi)$
BC:	$u_x(0,t) = u(\pi, t) = 0$	for $t > 0$
IC:	$u(x,0) = 3\cos\left(\frac{3}{2}x\right)$	for $x \in [0, \pi]$.

Note that in the BC we have $\frac{\partial u}{\partial x}(0,t) = 0$, and not $u(0,t) = 0$. Let $u(x,t)$ be a solution. Decide which of the following are true and which are false.

(i) $u(\frac{\pi}{3}, 0) = 0$	■ True	☐ False	☐ I don't know
(ii) $u(\frac{\pi}{6}, 1) = \frac{3\sqrt{2}}{2e^9}$	■ True	☐ False	☐ I don't know
(iii) $u(1, \frac{1}{2}) = \frac{3}{2}e^{-2}\sin\left(\frac{3}{2}\right)$	☐ True	■ False	☐ I don't know
(iv) $u(\frac{5}{6}\pi, \frac{1}{4}) = -\frac{3}{\sqrt{2}}e^{-\frac{9}{4}}$	■ True	☐ False	☐ I don't know

Solution:
Using the Ansatz $u(x,t) = X(x)T(t)$, we proceed as in the lecture to obtain

$$u(x,t) = \begin{cases} Ax + B \\ e^{4\lambda^2 t}(Ae^{\lambda x} + Be^{-\lambda x}) \\ e^{-4\lambda^2 t}(A\sin(\lambda x) + B\cos(\lambda x)). \end{cases}$$

And, deriving once the variable x, we have:

$$u_x(x,t) = \begin{cases} A \\ e^{4\lambda^2 t}(A\lambda e^{\lambda x} - B\lambda e^{-\lambda x}) \\ e^{-4\lambda^2 t}(A\lambda\cos(\lambda x) - B\lambda\sin(\lambda x)). \end{cases}$$

It is easily verified that matching the initial condition $u_x(0,t) = 0$ provides respectively

$$\begin{aligned} A &= 0 \\ A &= B \\ A &= 0. \end{aligned}$$

Now consider the condition $u(\pi,t) = 0$. Matching the condition (remembering our findings on A and B) provides

$$\begin{aligned} B &= 0 \\ A = B &= 0 \\ B = 0 \text{ or } \cos(\lambda\pi) &= 0 \leftrightarrow \lambda = \frac{2n+1}{2} \text{ for some } n. \end{aligned}$$

Thus, the only non-zero solution that we obtain is

$$u(x,t) = Be^{-4\left(\frac{2n+1}{2}\right)^2 t}\cos\left(\frac{2n+1}{2}x\right).$$

We want now to match the solution we found with our initial condition. Luckily, the initial condition is already of the same form of the solution obtained with the Ansatz $u(x,t) = X(x)T(t)$, choosing $B = 3$ and $n = 1$. Thus we have that the solution of the IBVP is

$$u(x,t) = 3e^{-9t}\cos\left(\frac{3}{2}x\right).$$