Analysis III (BAUG) **Facsimile Final Exam**

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The Final Exam will consist of 6 Questions. Four will be multiple choice questions, and 2 will be written. The following 6 questions are a facsimile of the exam. There are some differences, however. Firstly, none of the question is multiple choice, and secondly, we are not assuming that you will do the following in 2 hours. Please note that the last exercise is non-standard and may look scary. However, we highly encourage you to try to solve it.

Question 1:

Note! To give you more opportunity to practice, Question 1 consist of two exercises, that are both past exam questions. Of course, expect only one exercise in the final exam.

1. Consider the following ODE for $y(x)$ with $0 \leq x \leq 4$, where $\delta(x)$ is the Dirac delta function:

$$\begin{gathered} y''''(x) = -6\delta(x-2) - 6\delta(x-1) \\ y(0) = 0 \qquad y'(0) = 0 \\ y''(4) = 0 \qquad y'''(4) = 0 \end{gathered}$$

   (i) What does the above ODE describe?

   (ii) Solve the above ODE for $y(x)$.

Solution:

This ODE models a beam fixed at one end with point masses located at $x = 1$ and $x = 2$.

Applying the Laplace transform to this ODE, we get

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = s^4Y(s) - sy''(0) - y'''(0) = -6e^{-2s} - 6e^{-s}.$$

Let $A = y''(0)$ and $B = y'''(0)$, then $s^4Y(s) - As - B = -6e^{-2s} - 6e^{-s}$, hence

$$Y(s) = \frac{As + B - 6e^{-2s} - 6e^{-s}}{s^4} = \frac{A}{s^3} + \frac{B}{s^4} - \frac{6e^{-2s}}{s^4} - \frac{6e^{-s}}{s^4}.$$

Taking the Laplace inverse, we get

$$y(x) = \frac{A}{2}x^2 + \frac{B}{6}x^3 - (x-2)^3u(x-2) - (x-1)^3u(x-1).$$

This function must also satisfy the remaining two boundary conditions :

$$y''(4) = A + 4B - 6(4-2) - 6(4-1) = 0$$

$$y'''(4) = B - 6 - 6 = 0$$

These imply that $B = 12$ and $A = -18$. Thus the solution is:

$$y(x) = 2x^3 - 9x^2 - (x-2)^3 u(x-2) - (x-1)^3 u(x-1).$$

2. Consider the following ODE for $y(x)$ with $0 \leq x \leq 3$:

$$y''''(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1 \\ 24 & \text{for } 1 \leq x < 2 \\ 0 & \text{for } 2 \leq x \leq 3 \end{cases}$$

$$y(0) = 0 \qquad y''(0) = 0$$
$$y(3) = 0 \qquad y''(3) = 0$$

(i) What does the above ODE describe?

(ii) Solve the above ODE for $y(x)$.

Solution:

(i) This IVP describes a beam of length 3 (parametrized by $0 \leq x \leq 3$) simply supported at both ends, with $EI = 1$, and a force applied to the beam described by the function $f(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1 \\ 24 & \text{for } 1 \leq x < 2 \\ 0 & \text{for } 2 \leq x \leq 3 \end{cases}$

(ii) The differential equation can be rewritten in terms of the heaviside step function as follows:

$$y''''(x) = 24u(x-1) - 24u(x-2).$$

Taking Laplace transforms:

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = \frac{24e^{-s}}{s} - \frac{24e^{-2s}}{s}$$

Substituting in $y''(0) = 0$ and $y(0) = 0$ gives:

$$s^4 Y(s) - s^2 y'(0) - y'''(0) = \frac{24}{s}(e^{-s} - e^{-2s})$$

Rearranging gives:

$$Y(s) = \frac{24e^{-s}}{s^5} - \frac{24e^{-2s}}{s^5} + \frac{y'(0)}{s^2} + \frac{y'''(0)}{s^4}$$

Taking inverse Laplace transforms gives:

$$y(x) = (x-1)^4 u(x-1) - (x-2)^4 u(x-2) + y'(0)x + y'''(0)\frac{x^3}{3!}$$

Plugging in $y(3) = 0$ gives

$$0 = 2^4 - 1^4 + y'(0)3 + y'''(0)\frac{3^3}{3!}$$

Plugging in $y''(3) = 0$ gives

$$0 = 4 \cdot 3 \cdot 2^2 - 4 \cdot 3 + 0 + 3y'''(0)$$

This implies $y'''(0) = -12$ and $y'(0) = 13$

Thus the solution is

$$y(x) = (x-1)^4 u(x-1) - (x-2)^4 u(x-2) + 13x - 2x^3$$

Question 2:

Consider the following IVP for $u(x,t)$.

$$\begin{aligned} \text{PDE}: \quad & u_{tt}(x,t) = u_{xx}(x,t) \qquad \text{for } -\infty < x < \infty \text{ and } t > 0 \\ \text{IC}: \quad & u(x,0) = \begin{cases} \frac{1}{1+x} & \text{for } x \geq 0 \\ \frac{1}{1-x} & \text{otherwise} \end{cases} \\ & u_t(x,0) = \begin{cases} x^2 & \text{for } -1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(i) What is $u(3,2)$?

(ii) For fixed $x \in \mathbb{R}$ what is $\lim_{t\to\infty} u(x,t)$?

Solution:

Using d'Alembert's Formula, the general solution to the above PDE is

$$u(x,t) = \frac{1}{2}\left(f(x-t) + f(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy$$

for $f(x) = \begin{cases} \frac{1}{1+x} & \text{for } x \geq 0 \\ \frac{1}{1-x} & \text{otherwise} \end{cases}$ and $g(x) = \begin{cases} x^2 & \text{for } -1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$.

(i) Plugging in $x = 3, t = 2$ we have

$$\begin{aligned} u(3,2) &= \frac{1}{2}\left(f(1) + f(5)\right) + \frac{1}{2}\int_{1}^{5} g(y)dy \\ &= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{6}\right) + \frac{1}{2}\int_{1}^{2} x^2 dy \\ &= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{6}\right) + \frac{1}{2}\left[\frac{x^3}{3}\right]_1^2 \\ &= \frac{3}{2}. \end{aligned}$$

(ii) For $t > |x|$, we have $f(x-t) = \frac{1}{1-x+t}$ and $f(x+t) = 1/(1+x+t)$ and $\int_{x-t}^{x+t} g(y)dy = \int_{-1}^{3} x^3 dx = \left[\frac{x^2}{3}\right]_{-1}^{3} = \frac{8}{3}$. Plugging these in gives

$$\begin{aligned} \lim_{t\to\infty} u(x,t) &= \lim_{t\to\infty}\left[\frac{1}{2}\left(f(x-t) + f(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy\right] \\ \lim_{t\to\infty} u(x,t) &= \lim_{t\to\infty}\left[\frac{1}{2}\left(f(x-t) + f(x+t)\right) + \left[\frac{x^3}{3}\right]_{-1}^{2}\right] \\ &= \lim_{t\to\infty}\left[\frac{1}{2}\left(\frac{1}{1-x+t} + \frac{1}{1+x+t}\right) + \frac{3}{2}\right] = \frac{3}{2}. \end{aligned}$$

Question 3:

Solve the following:

$$y'' + 4y' + 5y = 100e^{-2t}$$

Assuming $y(0) = -1, y'(0) = 0$.

Solution:

Using the Laplace transform we obtain:

$$\begin{aligned}
&s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 5Y(s) = 100\frac{1}{s+2}\\
&Y(s)(s^2+4s+5) = -s - 4 + \frac{100}{s+2}\\
&Y(s)(s^2+4s+5) = \frac{-s^2-6s+92}{s+2}\\
&Y(s) = \frac{-s^2-6s+92}{(s+2)(s^2+4s+5)}
\end{aligned}$$

We observe that $s^2+4s+5 = (s+2)^2+1$, which is a denominator that appears in the Table of Laplace Transforms. We want to find A, B, C, D such that:

$$\frac{As+B}{s+2} + \frac{Cs+D}{(s+2)^2+1} = \frac{-s^2-6s+92}{(s+2)((s+2)^2+1)}.$$

We multiply and obtain

$$As^3 + 4As^2 + 5As + Bs^2 + 4Bs + 5B + Cs^2 + 2Cs + Ds + 2D = -s^2 - 6s + 92 \leftrightarrow$$

$$\leftrightarrow \begin{cases} A = 0 \\ B + C = -1 \\ 4B + 2C + D = -6 \\ 5B + 2D = 92 \end{cases}$$

Solving the above system of linear equations, we obtain $b = 100, C = -101, D = -204$.

Thus we have:

$$Y(s) = \frac{100}{s+2} - \frac{101s+204}{(s+2)^2+1}.$$

We observe that $204 = 2 \cdot 101 + 2$. Thus we have:

$$Y(s) = 100\frac{1}{s+2} - 101\frac{s+2}{(s+2)^2+1} - 2\frac{1}{(s+2)^2+1}.$$

Thus we obtain:

$$y(t) = 100e^{-2t} - 101e^{-2t}\cos(t) - 2e^{-2t}\sin(t) = e^{-2t}(100 - 101\cos(t) - 2\sin(t)).$$

Question 4:

Solve the following IBVP:

$$\begin{cases} u_t = u_{xx} & \text{in } \Omega = (-L, L) \times (0, \infty) \\ u(-L,t) = u(L,t) & \text{for all } t > 0 \\ u_x(-L,t) = u_x(L,t) & \text{for all } t > 0 \\ u(x,0) = 2L + \frac{3}{4}\sin\left(\frac{7\pi}{L}x\right) - e^3\cos\left(\frac{5\pi}{L}x\right) & \text{for all } x \in [-L, L] \end{cases}$$

Solution:
Since $u(x,0)$ is already developed as trigonometric sum, we get $a_0 = 2L$, $a_5 = -e^3$ and $b_7 = \frac{3}{4}$. Thus we apply the formula and get:

$$u(x,t) = 2L - e^{-\left(\frac{5\pi}{L}\right)^2 t + 3}\cos\left(\frac{5\pi}{L}x\right) + \frac{3}{4}e^{-\left(\frac{7\pi}{L}\right)^2 t}\sin\left(\frac{7\pi}{L}x\right).$$

Question 5:

Let $f\colon [-\pi, \pi] \to \mathbb{R}$ be the following function:

$$f(x) = \begin{cases} x + \pi & \text{if } x \in [-\pi, 0] \\ -x + \pi & \text{if } x \in [0, \pi] \end{cases}$$

Compute the Fourier series of f.

Solution:

Let

$$g(x) = \begin{cases} x & \text{if } x \in [-\pi, 0] \\ -x & \text{if } x \in [0, \pi] \end{cases}$$

It is clear that $f(x) = g(x) + \pi$. Thus we have:

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} g(x)dx + \int_{-\pi}^{\pi} \pi dx\right) = \frac{1}{2\pi}\left(\int_{-\pi}^{0} xdx - \int_{0}^{\pi} xdx + 2\pi^2\right) = \\ &= \frac{1}{2\pi}\left(-\frac{\pi^2}{2} - \frac{\pi^2}{2} + 2\pi^2\right) = \frac{\pi}{2}. \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\left(\int_{-\pi}^{\pi} g(x)\cos(nx)dx + \pi\int_{-\pi}^{\pi} \cos(nx)dx\right) = \\ &= \frac{1}{\pi}\left(\int_{-\pi}^{0} x\cos(nx)dx - \int_{0}^{\pi} x\cos(nx)dx + 0\right) = \\ &= \frac{1}{n\pi}\left(x\sin(nx)|_{-\pi}^{0} - \int_{-\pi}^{0} \sin(nx)dx - x\sin(nx)|_{0}^{\pi} + \int_{0}^{\pi} \sin(nx)dx\right) = \\ &= \frac{1}{n\pi}\left(\frac{1}{n}(1 - \cos(-n\pi)) - \frac{1}{n}(\cos(n\pi) - 1)\right) = \begin{cases} \frac{4}{n^2\pi} \text{ for } n \text{ odd} \\ 0 \text{ else} \end{cases} \end{aligned}$$

$$b_n = \frac{1}{\pi}\left(\int_{-\pi}^{\pi} g(x)\sin(nx)dx + \pi\int_{-\pi}^{\pi} \sin(nx)dx\right)$$

Observe that $g(x)$ is an even function, and $\sin(nx)$ is odd. Thus all the b_n terms are zero.

Question 6:

The goal of this exercise is to prove that the total energy of a vibrating string does not change with time. Let's assume that the string has length L and has fixed endpoints.

Moreover, we will ignore the effects of friction. That is, assume that the string is modeled by the following IBVP:

$$u_{tt} = u_{xx} \text{ for } (x,t) \in (0,L) \times (0,\infty)$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x)$$

$$u_t(x,0) = g(x).$$

We will will show that, regardless of the initial conditions f and g, the total energy of the system is constant.

Recall that the energy of the system at the time t is defined as:

$$H(t) = \frac{1}{2}\int_0^L u_t^2(x,t) + u_x^2(x,t)dx.$$

Indeed, we have that $\frac{1}{2}\int_0^L u_t^2(x,t)dx$ is the kinetic energy of the system, and $\frac{1}{2}\int_0^L u_x^2(x,t)dx$ is the potential energy. Intuitively, the first term measures how fast the speed of each single point will change after the time t. In particular this gives the kinetic energy that the point possesses. The integral should be just thought as the sum over all the points. The second term measures the potential energy. In order to see that imagine that every point x of the string is "pulled back" by a spring. We know that the potential energy for a spring of height y is $\frac{1}{2}ky^2$ and, for simplicity, we assume $k = 1$. Consider a point x_0. We need to understand what is y for x_0 and hence, where is going to be pulled back the point. Since the string is continuous, at least in the very near future the point x_0 is just going to be pulled back in the direction of the points that are very near to it! Intuitively, at least locally the string will "straighten up". That means that y is the difference of the height (at time t) of the point x_0 and the points around it, which is the derivative of u in the direction x.

Show that $H(t)$ is constant.

Solution:

To show that $H(t)$ is constant, we can show that its derivative is always zero. We obtain:

$$\begin{aligned}\frac{dH}{dt} &= \frac{1}{2}\int_0^L \frac{\partial}{\partial t}u_t^2(x,t) + \frac{\partial}{\partial t}u_x^2(x,t)dx = \\ &= \frac{1}{2}\int_0^L 2u_tu_{tt} + 2u_xu_{xt}dx\end{aligned}$$

Note that $u_{tt} = u_{xx}$! Integrating by parts the first term we obtain:

$$\begin{aligned} \frac{dH}{dt} &= \int_0^L u_t u_{xx} dx + \int_0^L u_x u_{xt} dx = \\ &= u_t u_x|_0^L - \int_0^L u_{tx} u_x dx + \int_0^L u_x u_{xt} dx = \\ &= u_t(0,t)u_x(0,t) - u_t(L,t)u_x(L,t). \end{aligned}$$

Since $u(0,t)$ and $u(L,t)$ are constant as functions of t, indeed, they are constantly zero, both of the terms above vanish. Thus the quantity H is constant in time.