Analysis III (BAUG) **Repetition**

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Question 1:

Solve the following IBVP:

$$\begin{cases} u_t = 25u_{xx} & \text{in } \Omega = (0,L)\times(0,\infty) \\ u_x(0,t) = u_x(L,t) = 0 & \text{for all } t>0 \\ u(x,0) = 6 + \cos\left(\frac{7\pi}{L}x\right) - 9\cos\left(\frac{9\pi}{L}x\right) & \text{for all } x\in[0,L] \end{cases}$$

Suppose that the solution $u(x,t)$ has the form:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} e^{c_n t}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right).$$

Decide if the following are true or false.

- $a_0 = 6$; **Yes**
- $a_7 = 1$; **Yes**
- $b_7 = 1$; **No, it is zero**
- $a_9 = 9$; **No, it is -9**
- $c_9 = -25\left(\frac{9\pi}{L}\right)^2$. **Yes**

Solution:
Since $u(x,0)$ is already developed as trigonometric sum, we get $a_0 = 6, a_7 = 1$ and $a_9 = -9$. Thus we apply the formula and get:

$$u(x,t) = 6 + e^{-25\left(\frac{7\pi}{L}\right)^2 t}\cos\left(\frac{7\pi}{L}x\right) - 9e^{-25\left(\frac{9\pi}{L}\right)^2 t}\cos\left(\frac{9\pi}{L}x\right).$$

Question 2:

Solve the following:

$$y'' + y = 1 - u\left(x - \frac{\pi}{2}\right)$$

assuming $y(0) = y'(0) = 0$.

Decide if the following are true or false.

- $y(0) = 0$; **Yes**
- $y(\pi) = 1$; **Yes**
- $y\left(\frac{3}{2}\pi\right) = 1$; **No, it is** -1
- $y(x)$ contains a term of the form cx^3, for some non-zero coefficient c; **No**
- If $x < \frac{\pi}{2}$, then $y(x) \geq 0$. **Yes**

Solution:
Taking the Laplace transform we obtain:

$$s^2Y(s) + sy(0) + y'(0) + Y(s) = \frac{1}{s} - \frac{e^{\frac{\pi}{2}s}}{s}$$
$$Y(s) = \frac{1}{s(s^2+1)} - \frac{e^{\frac{\pi}{2}s}}{s(s^2+1)}$$

Using the Laplace inverse we obtain:

$$y(x) = (1 - \cos(x)) - u\left(x - \frac{\pi}{2}\right)\left(1 - \cos\left(x - \frac{\pi}{2}\right)\right)$$

We have that $y(0) = 0$ by hypothesis. Substituting $x = \pi$ we obtain:

$$y(\pi) = (1+1) - 1 \cdot (1-0) = 1.$$

Substituting $x = \frac{3}{2}\pi$ we obtain:

$$y\left(\frac{3}{2}\pi\right) = (1-0) - 1 \cdot (1+1) = -1.$$

Note that for $x < \frac{\pi}{2}$ we have that $u(x - \frac{\pi}{2}) = 0$. Thus $y(x) = 1 - \cos(x)$. In particular $y(x) \geq 0$.

Question 3:

Consider the following IVP for $u(x,t)$.

$$\text{PDE}: \quad u_{tt}(x,t) = 4u_{xx}(x,t) \qquad \text{for } -\infty < x < \infty \text{ and } t > 0$$

$$\text{IC}: \quad u(x,0) = \begin{cases} \cos\left(\frac{2}{3}x\right) & \text{for } -\frac{3}{4}\pi \le x \le \frac{3}{4}\pi \\ 0 & \text{otherwise} \end{cases}$$

$$u_t(x,0) = \begin{cases} \cos(x) & \text{for } -\frac{3}{4}\pi \le x \le \frac{3}{4}\pi \\ 0 & \text{otherwise} \end{cases}$$

Decide if the following are true or false:

- $u(10,1) = \frac{1}{2}$; **No, it is 0**
- $u(\frac{\pi}{2}, \frac{\pi}{4}) = 0$; **No, it is** $\frac{2+\sqrt{2}}{8}$
- $u\left(0, \frac{3}{8}\pi\right) = \frac{\sqrt{2}}{4}$; **Yes**
- For fixed $x \in \mathbb{R}$ we have $\lim_{t\to\infty} u(x,t) = \frac{\sqrt{2}}{4}$. **Yes**

Solution:

Using d'Alembert's Formula, the general solution to the above PDE is

$$u(x,t) = \frac{1}{2}\left(f(x-2t) + f(x+2t)\right) + \frac{1}{4}\int_{x-2t}^{x+2t} g(y)dy$$

for $f(x) = \begin{cases} \cos\left(\frac{2}{3}x\right) & \text{for } -\frac{3}{4}\pi \le x \le \frac{3}{4}\pi \\ 0 & \text{otherwise} \end{cases}$ and $g(x) = \begin{cases} \cos(x) & \text{for } -\frac{3}{4}\pi \le x \le \frac{3}{4}\pi \\ 0 & \text{otherwise} \end{cases}$.

(i) Plugging in $x = 10, t = 1$ we have

$$u(10,1) = \frac{1}{2}\left(f(8) + f(12)\right) + \frac{1}{4}\int_{8}^{12} g(y)dy$$

Since $8 > \frac{3}{4}\pi$, we have that the functions f and g are constantly zero in the interval $[8, 12]$, yielding that $u(10,1) = 0$.

(ii) Plugging in $x = \frac{\pi}{2}$ and $t = \frac{\pi}{4}$ we have

$$u\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = \frac{1}{2}\left(\cos\left(\frac{2}{3}\cdot 0\right) + \cos\left(\frac{2}{3}\cdot\pi\right)\right) + \frac{1}{4}\int_0^{\frac{3}{4}\pi}\cos(y)dy =$$
$$= \frac{1}{2}\left(1 - \frac{1}{2}\right) + \frac{1}{4}\left[\sin(y)\right]_0^{\frac{3}{4}\pi} = \frac{1}{4} + \frac{\sqrt{2}}{8} = \frac{2+\sqrt{2}}{8}$$

(iii) Plugging in $x = 0$ and $t = \frac{3}{8}\pi$ we have

$$u\left(0, \frac{3}{8}\pi\right) = \frac{1}{2}\left(\cos\left(\frac{2}{3}\cdot\frac{3}{4}\pi\right) + \cos\left(\frac{2}{3}\cdot -\frac{3}{4}\pi\right)\right) + \frac{1}{4}\int_{-\frac{3}{4}\pi}^{\frac{3}{4}\pi} \cos(y)dy =$$
$$= 0 + \frac{1}{4}\left[\sin(y)\right]_{-\frac{3}{4}\pi}^{\frac{3}{4}\pi} = \frac{\sqrt{2}}{4}$$

(iv) For $|x + 2t|$ and $|x - 2t|$ large enough we have that $f(x + 2t) = f(x - 2t) = 0$. Moreover, for $a < -\frac{3}{4}\pi$ and $b > \frac{3}{4}\pi$ we have that $\int_a^b g(y)dy = \int_{-\frac{3}{4}\pi}^{\frac{3}{4}\pi} g(y)dy$. Thus we have:

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty}\left(\frac{1}{2}\left(f(x-2t) + f(x+2t)\right) + \frac{1}{4}\int_{x-2t}^{x+2t} g(y)dy\right) =$$
$$= \frac{1}{4}\int_{-\frac{3}{4}\pi}^{\frac{3}{4}\pi} g(y)dy = \frac{1}{4}\left[\sin(y)\right]_{-\frac{3}{4}\pi}^{\frac{3}{4}\pi} =$$
$$= \frac{\sqrt{2}}{4}$$

Question 4:

Suppose that we have a water tower (beam column) of length $\frac{9}{4}\pi$ (parametrized by $0 \le x \le \frac{9}{4}\pi$) and load $W = 5$ and $EI = 5$. Suppose that the wind applies a constant lateral force of intensity $f(x) = 10$. Let $y(x)$ be the deflection curve.

Hint: the fact that the water tower is a "vertical" beam does not effect the way the boundary conditions are determined.

Decide if the following are true or false:

- $y'(0) = 0$; **Yes**
- $y''(0) = \sqrt{2}$; **No, it is** $2 - \sqrt{2}$
- $y'''(0) = \sqrt{2} - 4\pi$; **No, it is** $\sqrt{2}$
- $y(2\pi) = 2\sqrt{2}\pi + 4\pi^2$; **No, it is** $4\pi^2 - 2\sqrt{2}\pi$
- $y(x)$ contains a term of the form cx^2, for some non-zero coefficient c. **Yes**

Solution:
The ODE that we want to solve is:

$$y''''(x) + \frac{5}{5}y''(x) = \frac{10}{5},$$

with boundary conditions $y(0) = y'(0) = y''\left(\frac{9}{4}\pi\right) = y\left(\frac{9}{4}\pi\right)''' = 0$.

Applying the Laplace transform to this ODE, we get

$$\begin{aligned} &s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) + s^2Y(s) - sy(0) - y'(0) = \frac{2}{s}, \\ &s^4Y(s) - sy''(0) - y'''(0) + s^2Y(s) = \frac{2}{s}. \end{aligned}$$

Let $A = y''(0)$ and $B = y'''(0)$. We have:

$$\begin{aligned} Y(s)\left(s^4 + s^2\right) &= As + B + \frac{2}{s}, \\ = Y(s) &= \frac{A}{s(s^2+1)} + \frac{B}{s^2(s^2+1)} + \frac{2}{s^3(s^2+1)}. \end{aligned}$$

Taking the Laplace inverse transform we obtain:

$$y(x) = A(1 - \cos(x)) + B(x - \sin(x)) + 2(-1 + \frac{x^2}{2} + \cos(x)).$$

We use the boundary conditions $y''\left(\frac{9}{4}\pi\right) = y'''\left(\frac{9}{4}\pi\right) = 0$ to get the values of A and B. We start by computing the derivatives of $y(x)$.

$$\begin{aligned} y'(x) &= A\sin(x) + B(1 - \cos(x)) + 2(x - \sin(x)); \\ y''(x) &= A\cos(x) + B\sin(x) + 2(1 - \cos(x)); \\ y'''(x) &= -A\sin(x) + B\cos(x) + 2\sin(x). \end{aligned}$$

Substituting $x = \frac{9}{4}\pi$ we obtain:

$$\begin{cases} y''\left(\frac{9}{4}\pi\right) = A\cos\left(\frac{9}{4}\pi\right) + B\sin\left(\frac{9}{4}\pi\right) - 2\cos\left(\frac{9}{4}\pi\right) + 2 = 0 \\ y'''\left(\frac{9}{4}\pi\right) = -A\sin\left(\frac{9}{4}\pi\right) + B\cos\left(\frac{9}{4}\pi\right) + 2\sin\left(\frac{9}{4}\pi\right) = 0 \end{cases}$$

$$\begin{cases} y''\left(\frac{9}{4}\pi\right) = A\frac{\sqrt{2}}{2} + B\frac{\sqrt{2}}{2} - 2\frac{\sqrt{2}}{2} + 2 = 0 \\ y'''\left(\frac{9}{4}\pi\right) = -A\frac{\sqrt{2}}{2} + B\frac{\sqrt{2}}{2} + 2\frac{\sqrt{2}}{2} = 0 \end{cases}$$

Dividing the second equation by $\frac{\sqrt{2}}{2}$, we obtain:

$$A = B + 2.$$

Substituting in the first equation, we have:

$$B\frac{\sqrt{2}}{2} + \sqrt{2} + B\frac{\sqrt{2}}{2} - \sqrt{2} + 2 = 0$$

and hence

$$B = -\sqrt{2}, \qquad A = 2 - \sqrt{2}.$$

Thus we obtain:

$$y(x) = \sqrt{2}\left(\cos(x) + \sin(x) + \frac{1}{\sqrt{2}}x^2 - x - 1\right).$$

Now we can compute $y(2\pi)$ substituting in the expression above.

$$\begin{aligned} y(2\pi) &= \sqrt{2}\left(1 + 0 + \frac{1}{\sqrt{2}}4\pi^2 - 2\pi - 1\right) = \\ &= 4\pi^2 - 2\sqrt{2}\pi \end{aligned}$$

Question 5:

Let $f\colon [-2,2] \to \mathbb{R}$ be the following function:

$$f(x) = \begin{cases} (x+1)^2 & \text{if } x \in [-2,-1] \\ 0 & \text{if } x \in [-1,1] \\ (x-1)^2 & \text{if } x \in [1,2] \end{cases}$$

Compute the Fourier series of f.

Solution:
We have:

$$\begin{aligned} a_0 &= \frac{1}{4}\left(\int_{-2}^{2} f(x)dx\right) = \frac{1}{4}\left(\int_{-2}^{-1}(x+1)^2dx + \int_{1}^{2}(x-1)^2dx\right) = \\ &= \frac{1}{4}\left(\left[\frac{(x+1)^3}{3}\right]_{-2}^{-1} + \left[\frac{(x-1)^3}{3}\right]_{1}^{2}\right) = \frac{1}{6} \end{aligned}$$

Observe that $f(x)$ is an even function. Thus, $f(x)\cos\left(\frac{n\pi}{2}x\right)$ is an even function as well, and $f(x)\sin\left(\frac{n\pi}{2}\right)$ is odd. In particular, we have that $b_n = 0$ for each n and that

$$\int_{-2}^{2} f(x)\cos\left(\frac{n\pi}{2}x\right)dx = 2\int_{0}^{2} f(x)\cos\left(\frac{n\pi}{2}x\right)dx.$$

Thus we obtain:

$$
\begin{aligned}
a_n &= \int_0^2 f(x)\cos\left(\frac{n\pi}{2}x\right)dx = \int_1^2 (x-1)^2\cos\left(\frac{n\pi}{2}x\right)dx = \\
&= \frac{2}{n\pi}\left[(x-1)^2\sin\left(\frac{n\pi}{2}x\right)\right]_1^2 - \frac{4}{n\pi}\int_1^2 (x-1)\sin\left(\frac{n\pi}{2}x\right)dx = \\
&= \frac{8}{n^2\pi^2}\left(\left[(x-1)\cos\left(\frac{n\pi}{2}x\right)\right]_1^2 - \int_1^2 \cos\left(\frac{n\pi}{2}x\right)dx\right) = \\
&= \frac{8}{n^2\pi^2}\left(\cos(n\pi) - \frac{2}{n\pi}\left[\sin\left(\frac{n\pi}{2}x\right)\right]_1^2\right) = \\
&= \frac{8}{n^2\pi^2}\left(\cos(n\pi) + \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)
\end{aligned}
$$

Question 6:

Consider a circular wire of length $2L$ (parametrized as $[-L, L]$) with initial temperature described by a function $\phi(x)\colon [-L, L] \to \mathbb{R}$. Show that the average heat of the rod does not change with time.
Hint! We recall that the solution $u(x,t)$ associated to the above IBVP represents the heat of the point x at the time t.

Solution:
We know that the behavior of the rod is described by an IBVP of type (3). That is, the function $u(x,t)$ that describes the temperature of the point x at the time t is the solution of the IBVP:

$$
\begin{cases}
u_t = \alpha^2 u_{xx} & \text{in } \Omega = (-L, L)\times(0,\infty) \\
u(-L,t) = u(L,t) & \text{for all } t > 0 \\
u_x(-L,t) = u_x(L,t) & \text{for all } t > 0 \\
u(x,0) = \phi(x) & \text{for all } x \in (-L, L).
\end{cases}
$$

for some coefficient α^2. The average temperature at the time t is given by:

$$
\mathrm{AT}(t) = \frac{1}{2L}\int_{-L}^{L} u(x,t)dx
$$

We show that the derivative of $\mathrm{AT}(t)$ is zero, that is that the function is constant in t.

$$\frac{d}{dt}\text{AT}(t) = \frac{1}{2L}\frac{d}{dt}\int_{-L}^{L} u(x,t)dx = \frac{1}{2L}\int_{-L}^{L} u_t(x,t)dx$$

Using $u_t = \alpha^2 u_{xx}$, we obtain:

$$\frac{d}{dt}\text{AT}(t) = \frac{\alpha^2}{2L}\int_{-L}^{L} u_{xx}(x,t)dx = \frac{\alpha^2}{2L}\, u_x(x,t)|_{-L}^{L} = \frac{\alpha^2}{2L}(u_x(-L,t) - u_x(L,t))$$

However, since we are considering a circular wire, the two above quantities are equal, thus their difference is zero.