Analysis III (BAUG) **Final exam**

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Question 1:

Suppose that we have a beam of length 6 (parametrized by $0 \le x \le 6$) simply supported at both ends. Suppose that $EI = 2$, and suppose that a force $F = 3$ is applied downwards at the point $x = 4$.

Let $y(x)$ be the deflection curve. Which of the following apply?

- ☐ $y'(0) = \frac{1}{2}$ **No: it is** $-\frac{8}{3}$
- ☐ $y(3) = -\frac{23}{4}$ **Yes**
- ☐ $y(x)$ contains a term of the form ax^3, for some non-zero coefficient a **Yes**
- ☐ $y''(6) = 1$ **No, it is one of the boundary conditions**
- ☐ $y(1) = -\frac{31}{12}$ **Yes**

Solution:

The ODE that we want to solve is:

$$2y''''(x) = -3\delta(x-4),$$

with boundary conditions $y(0) = y''(0) = y(6) = y''(6) = 0$.

Applying the Laplace transform to this ODE, we get

$$2s^4Y(s) - 2s^3y(0) - 2s^2y'(0) - 2sy''(0) - 2y'''(0) = -3e^{-4s}$$
$$2s^4Y(s) - 2s^2y'(0) - 2y'''(0) = -3e^{-4s}$$

Let $A = y'(0)$ and $B = y'''(0)$. We have:

$$Y(s) = \frac{-3e^{-4s} + 2As^2 + 2B}{2s^4} = \frac{-3e^{-4s}}{2s^4} + \frac{A}{s^2} + \frac{B}{s^4} =$$
$$= \frac{-1}{4}e^{-4s}\frac{6}{s^4} + A\frac{1}{s^2} + \frac{B}{6}\frac{6}{s^4}$$

Taking the Laplace inverse transform we obtain:

$$y(t) = -\frac{1}{4}(t-4)^3u(t-4) + At + \frac{B}{6}t^3.$$

We use the boundary conditions $y(6) = y''(6) = 0$ to get the values of A and B.

$$y(6) = -\frac{1}{4}(6-4)^3 + 6A + 36B = 0$$
$$y''(6) = -\frac{6}{4}(6-4) + 6B = 0$$

We obtain $B = \frac{1}{2}$ and $A = -\frac{8}{3}$.

Question 2:

Consider the following IVP for $u(x,t)$.

$$\begin{aligned} \text{PDE}: \quad & u_{tt}(x,t) = 4u_{xx}(x,t) \qquad \text{for } -\infty < x < \infty \text{ and } t > 0 \\ \text{IC}: \quad & u(x,0) = \begin{cases} 9 - x^2 & \text{for } -3 \le x \le 3 \\ 0 & \text{otherwise} \end{cases} \\ & u_t(x,0) = \begin{cases} (x+1)^2 - 4 & \text{for } -3 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Check all that apply:

- ☐ $u(-5,1) = \frac{1}{8}$ **No, it is zero**
- ☐ $u(3,2) = \frac{8}{3}$ **Yes**
- ☐ $u(0,1) = -4$ **No, it is $\frac{10}{3}$**
- ☐ $\lim_{t\to\infty} u(x,t) = -\frac{8}{3}$ **Yes**

Solution:
Using d'Alembert's Formula, the general solution to the above PDE is

$$u(x,t) = \frac{1}{2}\left(f(x-2t) + f(x+2t)\right) + \frac{1}{4}\int_{x-2t}^{x+2t} g(y)dy$$

for $f(x) = \begin{cases} 9 - x^2 & \text{for } -3 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$ and $g(x) = \begin{cases} (x+1)^2 - 4 & \text{for } -3 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$.

(i) We have that $f(-3) = f(-7) = 0$, and that g is constantly zero in the interval $[-7,-3]$. Thus $u(-5,1) = 0$.

(ii) Plugging in $x = 3, t = 2$ we have

$$
\begin{aligned}
u(3,2) &= \frac{1}{2}\left(f(-1) + f(7)\right) + \frac{1}{4}\int_{-1}^{7} g(y)dy \\
&= \frac{1}{2}\left(8 + 0\right) + \frac{1}{4}\int_{-1}^{1} (y+1)^2 - 4dy \\
&= 4 + \frac{1}{4}\left[\frac{(y+1)^3}{3} - 4y\right]_{-1}^{1} \\
&= 4 + \frac{1}{4}\left(\frac{8}{3} - 4 - 4\right) = 4 - \frac{4}{3} = \\
&= \frac{8}{3}
\end{aligned}
$$

(iii) Plugging in $x = 0, t = 1$ we have:

$$
\begin{aligned}
u(0,1) &= \frac{1}{2}\left(f(-2) + f(2)\right) + \frac{1}{4}\int_{-2}^{2} g(y)dy \\
&= \frac{1}{2}(9 - 4 + 9 - 4) + \frac{1}{4}\int_{-2}^{1} (y+1)^2 - 4dy = \\
&= 5 + \frac{1}{4}\left[\frac{(y+1)^3}{3} - 4y\right]_{-2}^{1} \\
&= 5 + \frac{1}{4}\left(\frac{8}{3} - 4 + \frac{8}{3} - 8\right) \\
&= 5 - \frac{5}{3} = \frac{10}{3}
\end{aligned}
$$

(iv) For $|x + 2t|$ and $|x - 2t|$ large enough we have that $f(x + 2t) = f(x - 2t) = 0$. Moreover, for $a < -3$ and $b > 1$ we have that $\int_a^b g(y)dy = \int_{-3}^1 g(y)dy$. Thus we have:

$$
\begin{aligned}
\lim_{t\to\infty} u(x,t) &= \lim_{t\to\infty}\left(\frac{1}{2}\left(f(x-2t) + f(x+2t)\right) + \frac{1}{4}\int_{x-2t}^{x+2t} g(y)dy\right) = \\
&= \frac{1}{4}\int_{-3}^{1} (y+1)^2 - 4dy = \frac{1}{4}\left[\frac{(y+1)^3}{3} - 4y\right]_{-3}^{1} = \\
&= \frac{1}{4}\left(\frac{8}{3} - 4 + \frac{8}{3} - 12\right) = -\frac{8}{3}
\end{aligned}
$$

Question 3:

Let $f\colon [-2,2] \to \mathbb{R}$ be the following function:

$$f(x) = \begin{cases} -x-1 & \text{if } x \in [-2,-1] \\ 0 & \text{if } x \in [-1,1] \\ x-1 & \text{if } x \in [1,2] \end{cases}$$

Recall that the Fourier series of f has the form:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{2}x\right) + b_n \sin\left(\frac{n\pi}{2}x\right)\right).$$

Decide if the following are true or false. *Hint! Is f even or odd?*

- ☐ $a_0 = -\frac{1}{6}$; **No, it is** $\frac{1}{4}$
- ☐ $a_2 = -\frac{4}{\pi^2}$; **No, it is** $\frac{2}{\pi^2}$
- ☐ $a_3 = -\frac{4}{9\pi^2}$; **Yes**
- ☐ $b_3 = 0$; **Yes**
- ☐ $b_2 = \frac{2}{\pi^2}$. **No, it is zero**

Solution:
We have:

$$a_0 = \frac{1}{4}\left(\int_{-2}^{2} f(x)dx\right) = \frac{1}{4}\left(-\int_{-2}^{-1}(x+1)dx + \int_{1}^{2}(x-1)dx\right) =$$
$$= \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{4}$$

Note that f is an even function. Thus, $\int_{-2}^{2} f(x)dx = 2\int_0^2 f(x)dx$.

$$
\begin{aligned}
a_n &= \frac{1}{2}\left(\int_{-2}^{2} f(x)\cos\left(\frac{n\pi}{2}x\right)dx\right) = \\
&= \int_{0}^{2} f(x)\cos\left(\frac{n\pi}{2}x\right)dx = \\
&= \int_{1}^{2} (x-1)\cos\left(\frac{n\pi}{2}x\right)dx = \\
&= \left[(x-1)\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}x\right)\right]_1^2 - \int_1^2 \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}x\right)dx = \\
&= \frac{4}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}x\right)\right]_1^2 = \\
&= \frac{4}{n^2\pi^2}\left(\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right).
\end{aligned}
$$

For the b_n, notice that $f(x)$ is an even function. Since the product of an even function with an odd function is odd, we have that

$$\int_{-2}^{2} f(x)\sin\left(\frac{n\pi}{2}x\right)dx = 0$$

for each n. Thus all the b_n are zero.

Question 4:

Solve the following IBVP:

$$
\begin{cases}
u_t = 9u_{xx} & \text{in } \Omega = (0,L)\times(0,\infty) \\
u(0,t) = u(L,t) = 0 & \text{for all } t > 0 \\
u(x,0) = \frac{7}{8}\sin\left(\frac{4\pi}{L}x\right) + \cos(\pi)\sin\left(\frac{6\pi}{L}x\right) & \text{for all } x \in [-L,L]
\end{cases}
$$

Suppose that the solution $u(x,t)$ has the form:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} e^{c_n t}\left(a_n\cos\left(\frac{n\pi}{L}x\right) + b_n\sin\left(\frac{n\pi}{L}x\right)\right).$$

Check all that apply.

- ☐ $a_0 = 0$ **Yes**
- ☐ $a_4 = \frac{3}{2}$ **No, it is zero**

☐ $b_5 = -1$ **No, it is zero**

☐ $b_4 = \frac{7}{8}$ **Yes**

☐ $c_6 = -\left(\frac{6\pi}{L}\right)^2$ **No, it is** $-9\left(\frac{6\pi}{L}\right)^2$

Solution:
Since $u(x,0)$ is already developed as trigonometric sum, we get $b_4 = \frac{7}{8}$ and $b_6 = -1$. Thus we apply the formula and get:

$$u(x,t) = \frac{7}{8}e^{-9\left(\frac{4\pi}{L}\right)^2 t}\sin\left(\frac{4\pi}{L}x\right) - e^{-9\left(\frac{6\pi}{L}\right)^2 t}\sin\left(\frac{6\pi}{L}x\right).$$

Question 5:

Solve the following:

$$y'' + 6y' + 5y = -16e^{-3t}$$

Assuming $y(0) = 9$, $y'(0) = -27$.

Solution:
Using the Laplace transform we obtain:

$$\begin{aligned}
s^2Y(s) - sy(0) - y'(0) + 6sY(s) - 6y(0) + 5Y(s) &= -52\frac{1}{s+3} \\
s^2Y(s) - 9s + 27 + 6sY(s) - 54 + 5Y(s) &= \frac{-16}{s+3} \\
Y(s)(s^2+6s+5) &= \frac{-16}{s+3} + 9s + 27 \\
Y(s)(s^2+6s+5) &= \frac{-16 + 9s^2 + 27s + 27s + 81}{s+3} \\
Y(s) &= \frac{9s^2 + 54s + 65}{(s+3)(s^2+6s+5)}
\end{aligned}$$

We observe that $s^2 + 6s + 5 = (s+3)^2 - 2^2$, which is a denominator that appears in the Table of Laplace Transforms. We want to find A, B, C, D such that:

$$\frac{As+B}{s+3} + \frac{Cs+D}{(s+3)^2-4} = \frac{9s^2+54s+65}{(s+3)((s+3)^2-4)}.$$

We multiply and obtain

$$As^3 + 6As^2 + 5As + Bs^2 + 6Bs + 5B + Cs^2 + 3Cs + Ds + 3D = 9s^2 + 54s + 65 \leftrightarrow$$

$$\leftrightarrow \begin{cases} A = 0 \\ B + C = 9 \\ 6B + 3C + D = 54 \\ 5B + 3D = 65 \end{cases}$$

Solving the above system of linear equations, we obtain $B = 4, C = 5, D = 15$.

Thus we have:

$$Y(s) = \frac{4}{s+3} + \frac{5s+15}{(s+3)^2 - 2^2}.$$

We observe that $5s + 15 = 5(s+3)$. Thus we have:

$$Y(s) = 4\frac{1}{s+3} + 5\frac{s+3}{(s+3)^2 - 2^2}.$$

We obtain:

$$y(t) = 4e^{-3t} + 5e^{-3t}\cosh(2t) = e^{-3t}(4 + 5\cosh(2t)).$$

Note that, since $\cosh(2t) = \frac{1}{2}\left(e^{2t} + e^{-2t}\right)$, the result can be also written as:

$$\begin{aligned} y(t) &= 4e^{-3t} + \frac{5}{2}e^{-3t}e^{2t} + \frac{5}{2}e^{-3t}e^{-2t} = \\ &= 4e^{-3t} + \frac{5}{2}e^{-t} + \frac{5}{2}e^{-5t}. \end{aligned}$$

Question 6:

Consider a completely insulated rod of length L (parametrized as $[0, L]$) with initial temperature described by a function $\phi(x)\colon [0, L] \to \mathbb{R}$. Show that the average temperature of the rod does not change with time.

Solution:

We know that the behavior of the rod is described by an IBVP of type (2). That is, the function $u(x, t)$ that describes the temperature of the point x at the time t is the solution of the IBVP:

$$\begin{cases} u_t = \alpha^2 u_{xx} & \text{in } \Omega = (0, L) \times (0, \infty) \\ u_x(0, t) = u_x(L, t) = 0 & \text{for all } t > 0 \\ u(x, 0) = \phi(x) & \text{for all } x \in [0, L] \end{cases}$$

for some coefficient α^2. The average temperature at the time t is given by:

$$\mathrm{AT}(t) = \frac{1}{L}\int_0^L u(x,t)dx$$

We show that the derivative of $\mathrm{AT}(t)$ is zero, that is that the function is constant in t.

$$\frac{d}{dt}\mathrm{AT}(t) = \frac{1}{L}\frac{d}{dt}\int_0^L u(x,t)dx = \frac{1}{L}\int_0^L u_t(x,t)dx$$

Using $u_t = \alpha^2 u_{xx}$, we obtain:

$$\frac{d}{dt}\mathrm{AT}(t) = \frac{\alpha^2}{L}\int_0^L u_{xx}(x,t)dx = \frac{\alpha^2}{L}\, u_x(x,t)|_0^L = \frac{\alpha^2}{L}(u_x(0,t) - u_x(L,t))$$

However, since we are considering a totally insulated rod, the boundary conditions imply that the two quantities above are constantly zero.