Analysis III for D-BAUG, Fall 2018 — Lecture 1

Lecturer: Alex Sisto (sisto@math.ethz.ch)*

1 What is a PDE?

Recall that an ordinary differential equation (ODE) is an equation where an unknown function of one variable, $u = u(x)$, is related to its derivatives u', u'', etc. For example,

$$u' = u. \tag{1.1}$$

A **partial differential equation (PDE)** is similar, but now the unknown function $u = u(x, y, \dots)$ depends on several variables, and the equation relates u to its partial derivatives. For example,

$$u_{xy}u_z + u_{tt} = 0,$$

where $u = u(t, x, y, x)$ depends on four variables. Here we used:

Notation: For a function $u = u(x_1, x_2, \dots, x_n)$ of n variables, the partial derivatives are denoted by

$$u_{x_i} = \frac{\partial u}{\partial x_i}, \qquad u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \qquad \text{etc.}$$

Remark 1.1. (i) Of course u needs to be sufficiently differentiable. In this course this will always be the case, and we will not give it much explicit mentioning.

(ii) Sometimes we only require that $u = u(x_1, \dots, x_n)$ satisfies the PDE in some open subset $\Omega \subset \mathbb{R}^n$. In such cases u might not be defined (or differentiable) outside Ω. □

2 Examples of PDE

The simplest possible PDE (that is not an ODE) is arguably $u_x = 0$, where $u = u(x, y)$ is a function of two variables. The solution is $u(x, y) = f(y)$, where $f(y)$ is an arbitrary function of y. Below are some more interesting examples, where the function $u = u(t, x)$ is understood to depend on two variables. The letter c denotes some given constant.

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(i) $u_t = u_{xx}$ (1-dimensional heat equation)

(ii) $u_t + uu_x = cu_{xx}$ (Burger's equation)

(iii) $u_{tt} = c^2 u_{xx}$ (1-dimensional wave equation)

(iv) $u_{tt} = -c^2 u_{xxxx}$ (beam equation)

(v) In general: $F(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots) = 0$, where F is a given function.

The variables t, x, y, etc. are called **independent variables**, and the function u is called the **dependent variable**. It might seem hard to believe, but these equations give detailed descriptions of a number of rather complex physical situations. Their solutions have very different properties.

3 Examples of solutions

Consider the 1-dimensional heat equation $u_t = u_{xx}$. Here are some examples of solutions of this equation:

(i) $u(t, x) = 0$ is obviously a solution.

(ii) $u(t, x) = \frac{1}{2}x^2 + t$ is a solution, because $u_t = 1 = u_{xx}$.

(iii) Consider functions of the form $u(t, x) = e^{ax+bt}$, where a and b are constants. For which values of a and b is this a solution?

Answer: Compute $u_t = be^{ax+bt}$ and $u_{xx} = a^2 e^{ax+bt}$. Therefore $u_t = u_{xx}$, i.e. u is a solution, if and only if $b = a^2$.

Remark 3.1. A guess for a general form of the solution, such as $u(t, x) = a^{ax+bt}$, is called an Ansatz (in English too!). Another very useful Ansatz that we will use many times is $u(t, x) = T(t)X(x)$ for functions $T(t)$ and $X(x)$ that are to be determined.

Depending on the situation, some Ansatzes work better than others. How does one know which one to try? Practice and experience! □

(iv) $u(t, x) = \frac{1}{2}x^2 + t + e^{x+t}$ is a solution. There are two ways to see this. Either compute u_t and u_{xx} directly, and check that they are equal. Or, make the following general observation: If v and w are two solutions, then so is $u = v + w$. Indeed,

$$u_t = (v + w)_t = v_t + w_t = v_{xx} + w_{xx} = (v + w)_{xx} = u_{xx}.$$

We can now re-use the results in (ii) and (iii) above.

The second approach was perhaps not very efficient in this particular case. However, it will turn out to be extremely important later on!

Remark 3.2. We have seen that a PDE can have lots of different solutions. To pin down a unique solution, further restrictions are needed. This will come in the form of boundary conditions. Compare this to the case of ODEs: for any choice of constant c, a solution of (1.1) is given by

$$u(x) = ce^x.$$

Selecting one specific solution boils down to specifying the value of $u(0) = c$. □

4 Classification of PDE

PDEs are classified in different groups depending on their characteristics. This reflects the fact that different methods are often required to analyze and solve different PDEs.

Order: The order of the highest derivative appearing in the PDE.

Number of variables: Number of independent variables.

Linear/nonlinear: The PDE is **linear** if u and its derivatives appear in a linear way, that is to say both sides are sums of terms of the form $F(x, y, ...)$ times u or a derivative of u (e.g. u_x, u_{xy}, $\dots$). Note that the independent variables (t, x, y, etc.) do not have to enter linearly. If the PDE is not linear, then it is **nonlinear**.

Example 4.1. The PDEs (i)–(iv) listed in Section 2 all have two variables (t and x). Furthermore,

(i) $u_t = u_{xx}$ is second-order and linear.

(ii) $u_t + uu_x = cu_{xx}$ is second-order nonlinear.

(iii) $u_{tt} = c^2 u_{xx}$ is second-order linear.

(iv) $u_{tt} = -c^2 u_{xxxx}$ is fourth-order linear.

Let's consider two more examples: The 3-dimensional wave equation, $u_{tt} = u_{xx} + u_{yy} + u_{zz}$ is a second-order linear PDE in four variables. The PDE $u_x^4 + u_y u^{18} = 0$ has two variables, is nonlinear, and of first order (don't get confused by the powers 4 and 18; the order of a PDE has nothing to do with the degree of polynomials!). □

Example 4.2. One easily gets confused about linearity vs. nonlinearity of PDEs before getting used to it. Here are some further examples:

(i) $u_{tt} = e^{-t} u_{xx} + \sin t$ is **linear**, since the only quantity that enter nonlinearly is the independent variable t.

(ii) $uu_{xx} + u_t = 0$ is **nonlinear** due to the product uu_{xx}.

(iii) $u_{xx} + yu_{yy} = 0$ is **linear**.

(iv) $xu_x + yu_y + u^2 = 0$ is **nonlinear** due to the term u^2. □

The following class of PDEs is the main focus of this course:

A **second-order linear PDE in two variables** is of the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \tag{4.1}$$

where the **coefficients** A, B, C, D, E, F, G may be functions of the independent variables x, y. Sometimes the independent variables will not be called x, y, but rather t, x (or something else).

Second-order linear PDEs are further classified into subgroups based on the properties of its coefficients. The PDE (4.1) is said to

- have **constant coefficients** if A, B, C, D, E, F, G are all constant. Otherwise, it is said to have **non-constant coefficients**.
- be **homogeneous** if $G = 0$. Otherwise, it is said to be **non-homogeneous**.
- have a **type** that is determined by the quantity $B^2 - 4AC$ according to the following scheme:

$B^2 - 4AC$	Type
$= 0$	Parabolic
> 0	Hyperbolic
< 0	Elliptic

Remark 4.3. (i) The type does not depend on the coefficients D, E, F, G.

(ii) If A, B, C depend on x and/or y (non-constant coefficients), then the type can be different for different values of x, y.

(iii) The type is invariant under a linear change of variables (more about this later!)

(iv) Why is the type relevant? It turns out that PDEs of the same type have similar features and can often be solved using similar methods. □

Example 4.4. Let's classify the second-order linear PDE $u_{tt} = e^{-t}u_{xx} + \sin t$ from Example 4.2(i). To do this, first move all the terms involving u to the left-hand side,

$$u_{tt} - e^{-t}u_{xx} = \sin t.$$

We can now read off the coefficients: $A = 1$, $B = 0$, $C = -e^{-t}$, $D = E = F = 0$, and $G = \sin t$. Clearly the coefficients are non-constant. The PDE is non-homogeneous since G is not zero. Finally, we have

$$B^2 - 4AC = 4e^{-t} > 0,$$

so the the PDE is hyperbolic.

Notice that in this example, the independent variables (t, x) corresponded to the variables (x, y) in (4.1). We could also have swapped the order between t and x, which would have led to $A = -e^{-t}$ and $C = 1$, and the remaining coefficients still zero. Fortunately, this does not affect $B^2 - 4AC$, and therefore also not the type. □

Example 4.5. Let's find the type of the Euler-Tricomi equation (used in the study of transonic flow),

$$yu_{xx} + u_{yy} = 0.$$

Here $A = y$, $B = 0$, $C = 1$, $D = E = F = G = 0$. Hence $B^2 - 4AC = -4y$, so the type depends on y. Specifically, the PDE is parabolic at $y = 0$, hyperbolic at $y < 0$, and elliptic at $y > 0$. □

Example 4.6. We end with three key examples of second-order linear homogeneous PDE with constant coefficients. Two of them we have already seen above. Most of this course will be about finding solutions to these equations!

> **1-dimensional heat equation:** $u_t = cu_{xx}$, where $c > 0$ is a constant. This equation is parabolic.
>
> **1-dimensional wave equation:** $u_{tt} = c^2 u_{xx}$, where c is a constant. This equation is hyperbolic.
>
> **2-dimensional Laplace equation:** $u_{xx} + u_{yy} = 0$. This equation is elliptic.

Why are linear PDEs often much nicer to work with than nonlinear PDEs? One important reason is the following proposition. Its proof uses the same kind of calculation as (iv) in Section 3, namely that the partial derivative of a sum equals the sum of the partial derivatives.

Proposition 4.7 (Superposition principle, first version)**.** *Let u and v be two solutions of the homogeneous second-order linear PDE* (4.1) *(we assume that $G = 0$ to make it homogeneous), and let a and b be two constants. Then $au + bv$ is another solution of the same PDE.*