Analysis III for D-BAUG, Fall 2018 — Lecture 10

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1 Convolution (*Faltung*)

We have already seen that the Laplace transform is not multiplicative, that is, $\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$ in general. By replacing the product on the left-hand side with a different operation, one can nonetheless obtain a similar identity which is useful for solving ODEs.

Definition. The **convolution** of $f(t)$ and $g(t)$ is the function

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau, \qquad t \geq 0.$$

Example 1.1. (i) Suppose $f(t) = e^{-t}$ and $g(t) = t$. Then

$$(f * g)(t) = \int_0^t e^{-\tau}(t-\tau)d\tau = e^{-t} + t - 1,$$

where the integral is computed using integration by parts. The details are left as an exercise.

(ii) Suppose $f(t) = \sin(\omega t)$ and $g(t) = \cos(\omega t)$. Then, making use of the trigonometric identity $\sin(\varphi)\cos(\theta) = \frac{1}{2}\sin(\theta+\varphi) - \frac{1}{2}\sin(\theta-\varphi)$, we compute

$$\begin{aligned}(f * g)(t) &= \int_0^t \sin(\omega\tau)\cos(\omega(t-\tau))d\tau \\ &= \frac{1}{2}\int_0^t \sin(\omega t)d\tau - \frac{1}{2}\int_0^t \sin(\omega t - 2\omega\tau)d\tau \\ &= \frac{1}{2}t\sin(\omega t) + \frac{1}{2}\frac{\cos(\omega t - 2\omega\tau)}{-2\omega}\Big|_0^t \\ &= \frac{1}{2}t\sin(\omega t).\end{aligned}$$

In many ways, but not all, convolution behaves like multiplication.

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Exercise 1.2. Check that the following identities are correct, where f, g, h are functions and c is a constant:

$$f * g = g * f, \quad f * (g + h) = f * g + f * h, \quad f * (cg) = c \cdot (f * g), \quad (f * g) * h = f * (g * h).$$

On the other hand, check that $f * 1 \neq f$ and $f * f \ngeq 0$ in general, by finding some explicit functions $f(t)$ that illustrate this.

Theorem. The convolution and the Laplace transform are related through the following identity:

$$\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}.$$

A useful equivalent form is

$$(f * g)(t) = \mathcal{L}^{-1}\{F(s)G(s)\}, \qquad \text{where } F(s) = \mathcal{L}\{f(t)\} \text{ and } G(s) = \mathcal{L}\{g(t)\}.$$

To verify the theorem we have to do a calculation which in principle is straightforward. It is still a little bit tricky however, because it involves double integrals and at one point we have to exchange the order of integration. Here it is:

$$\begin{aligned}
\mathcal{L}\{(f * g)(t)\} &= \int_0^\infty e^{-st}\left(\int_0^t f(\tau)g(t-\tau)d\tau\right)dt && \text{(by definition of convolution)}\\
&= \int_0^\infty \left(\int_0^t e^{-st} f(\tau)g(t-\tau)d\tau\right)dt && \text{(taking } e^{-st} \text{ inside the inner integral)}\\
&= \int_0^\infty \left(\int_\tau^\infty e^{-st} f(\tau)g(t-\tau)dt\right)d\tau && \text{(swapping order of integration)}\\
&= \int_0^\infty \left(\int_0^\infty e^{-s(u+\tau)} f(\tau)g(u)du\right)d\tau && \text{(changing variable to } u = t-\tau)\\
&= \int_0^\infty e^{-s\tau} f(\tau)\left(\int_0^\infty e^{-su} g(u)du\right)d\tau\\
&= \int_0^\infty e^{-s\tau} f(\tau)d\tau \int_0^\infty e^{-su} g(u)du\\
&= \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}.
\end{aligned}$$

The convolution theorem is yet another useful tool for computing inverse Laplace transform (last time we also looked at partial fractions, s-shifting, and t-shifting, and of course the linearity property is constantly being used).

Example 1.3. Let's compute the inverse Laplace transform of $F(s) = \frac{1}{(s+1)s^2}$. The convolution theorem gives

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^{-t} * t = e^{-t} + t - 1,$$

where we re-used the computation from Example 1.1(i).

2 Resonance in mass-spring systems

Last time we considered a special case of a mass-spring system described by

$$mf''(t) + cf'(t) + kf(t) = g(t), \tag{2.1}$$

where m represents the mass, c is a damping coefficient, k is a spring constant, and $g(t)$ represents external forcing. We will now study resonance in a simplified version of this system, namely

$$f''(t) + \omega_0^2 f(t) = g(t), \qquad f(0) = c_1,\ f'(0) = c_2 \tag{2.2}$$

for some constants c_1, c_2.

Exercise 2.1. If there is no external forcing, $g(t) = 0$, check that the solution of (2.2) is

$$f(t) = c_1 \cos(\omega_0 t) + \frac{c_2}{\omega_0} \sin(\omega_0 t).$$

In particular, the system oscillates at the frequency ω_0, which does not depend on the initial conditions c_1 and c_2.

Let's now consider the case where $c_1 = c_2 = 0$, but the forcing $g(t)$ is general. Taking the Laplace transform in (2.2) gives

$$s^2 F(s) + \omega_0^2 F(s) = G(s)$$

where $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$. Solve this to get

$$F(s) = \frac{1}{s^2 + \omega_0^2} G(s).$$

The convolution theorem and the fact that $\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega_0^2}\right\} = \frac{\sin(\omega_0 t)}{\omega_0}$ lead to the solution of the ODE,

$$f(t) = \int_0^t \frac{\sin(\omega_0 \tau)}{\omega_0} g(t - \tau) d\tau.$$

We have therefore obtained an explicit solution formula for a general forcing term $g(t)$, and we can now plug in various interesting choices for $g(t)$. In particular, suppose $g(t) = \cos(\eta t)$ which oscillates with some frequency η. If we take $\eta = \omega_0$, we can use Example 1.1(ii) to get the solution

$$f(t) = \frac{1}{2\omega_0} t \sin(\omega_0 t).$$

This solution has a peculiar form, namely that the amplitude of the oscillations grows unboundedly $t \to \infty$. This is the mathematical manifestation of **resonance**, where a dramatic reinforcement occurs if the frequency of the forcing is the same as the intrinsic frequency of the system. Of course, in reality the amplitude is not unbounded; instead one soon reaches a point where things like friction or structural limitations start playing a large role. At that point the ODE (2.2) is no longer an accurate description of how the system behaves.

Exercise 2.2. Suppose that $\eta \neq \omega_0$, and show that the solution of (2.2) is

$$f(t) = \frac{1}{\omega_0^2 - \eta^2} \left(\cos(\omega_0 t) - \cos(\eta t)\right).$$

How does the solution behave when η is very close to ω_0?

3 Volterra integral equations

Consider the equation

$$f(t) = g(t) + \int_0^t f(\tau)h(t-\tau)d\tau, \tag{3.1}$$

where $g(t)$ and $h(t)$ are given functions and $f(t)$ is an unknown function which is to be determined. Such equations are called **Volterra integral equations**, and while they do not play a big role in this course, they form a very important class of equations.

Note that the right-hand side of (3.1) is nothing but the convolution $(f * h)(t)$. This suggests that we can use the Laplace transform to solve the equation. Indeed, take the Laplace transform of both sides and use the convolution theorem to get

$$F(s) = G(s) + F(s)H(s),$$

where of course $F(s) = \mathcal{L}\{f(t)\}$, $G(s) = \mathcal{L}\{g(t)\}$, and $H(s) = \mathcal{L}\{h(t)\}$. The solution is therefore

$$f(t) = \mathcal{L}^{-1}\left\{\frac{G(s)}{1-H(s)}\right\}.$$

Example 3.1. Consider the following Volterra integral equation:

$$f(t) = e^{-t} + \int_0^t f(\tau)\sinh(t-\tau)d\tau.$$

Here $g(t) = e^{-t}$ and $h(t) = \sinh(t)$, and therefore using the table we find $G(s) = \frac{1}{s+1}$ and $H(s) = \frac{1}{s^2-1}$. Hence

$$F(s) = \frac{s}{s^2-2} - \frac{1}{s^2-2}.$$

Taking the inverse Laplace transform using the table gives the solution

$$f(t) = \cosh(\sqrt{2}t) - \frac{1}{\sqrt{2}}\sinh(\sqrt{2}t).$$

4 The Dirac delta function

What is the response of the mass-spring system (2.1) if $g(t)$ is a very strong force applied over a very short amount of time? (This is known as "impulse response" or "hammer blow response"). What happens to a beam when pressure is applied to it at one single point on the beam? (Next time we will look at the mathematics of beams).

To answer these questions we need the Dirac delta function. It captures the idea of an "infinitely large force applied over an infinitely short time period" or "infinite pressure applied over an infinitely small area". Mathematically, this corresponds to finding a function $\delta(t-a)$ with the property

$$\int_0^\infty f(t)\delta(t-a)dt = f(a) \tag{4.1}$$

for any function $f(t)$. Look at (4.1) and think about what it says. On the one hand, if $f(a) = 0$ then the integral in (4.1) is zero. This means that $\delta(t-a) = 0$ for $t \neq a$. On the other hand, if you plug in $f(t) = 1$, then (4.1) says that the "total area under the graph" is one. In other words, the graph of $\delta(t-a)$ is something like an infinitely narrow and infinitely tall spike, whose total area is equal to one. This is strange! And strictly speaking, no such function exists. However, it is still possible to define an object $\delta(t-a)$ by the requirement that (4.1) should hold. This is the **Dirac delta function**, even though, confusingly, it is not a function (it's what mathematicians call a probability distribution). Property (4.1) is called the **sifting property**, because $\delta(t-a)$ "sifts" out the value of $f(t)$ at the point $t = a$ (to sift = sieben, a sieve = ein Sieb).

We can understand the Dirac delta function as a limit in the following way. Consider functions

$$g_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}, & 0 \leq t < \varepsilon \\ 0, & \text{otherwise.} \end{cases}$$

The graph of $g_\varepsilon(t)$ (and $g_\varepsilon(t-a)$) is a thin rectangle of width ε, which however is very tall: its height is $\frac{1}{\varepsilon}$. Moreover, the area under the graph is one. Then, if $f(t)$ is a continuous function,

$$\int_0^\infty f(t) g_\varepsilon(t-a) dt = \frac{1}{\varepsilon} \int_a^{a+\varepsilon} f(t) \to f(a)$$

by the mean value theorem. Therefore, we can think of $\delta(t)$ as the limit

$$\delta(t) = \lim_{\varepsilon \to 0} g_\varepsilon(t).$$

Exercise 4.1. Check that $\delta(t-a)$ is the derivative of the shifted Heaviside function $u(t-a)$. Part of the exercise is to make sense of what it means for $\delta(t-a)$ to be this derivative.

It is extremely easy to compute the Laplace transform of $\delta(t-a)$, because the sifting property directly gives

$$\mathcal{L}\{\delta(t-a)\} = \int_0^\infty e^{-st} \delta(t-a) dt = e^{-as}.$$

Example 4.2. Let's look at the impulse response of a mass-spring system. More specifically, let's solve the ODE

$$f''(t) + 3f'(t) + 2f(t) = \delta(t-1), \qquad f(0) = f'(0) = 0.$$

Taking the Laplace transform gives

$$s^2 F(s) + 3sF(s) + 2F(s) = e^{-s},$$

and solving for $F(s)$ we get

$$F(s) = \frac{1}{s^2 + 3s + 2} e^{-s} = \frac{1}{s+1} e^{-s} - \frac{1}{s+2} e^{-s},$$

where the second equality can be derived using partial fractions. This is left as an exercise. Now, using the t-shifting property and the table of Laplace transforms, it follows that

$$\frac{1}{s+1} e^{-s} = \mathcal{L}\left\{e^{-t}\right\} e^{-s} = \mathcal{L}\left\{e^{-(t-1)} u(t-1)\right\} \quad \text{and}$$
$$\frac{1}{s+2} e^{-s} = \mathcal{L}\left\{e^{-2t}\right\} e^{-s} = \mathcal{L}\left\{e^{-2(t-1)} u(t-1)\right\},$$

and therefore

$$F(s) = \mathcal{L}\left\{e^{-(t-1)}u(t-1) - e^{-2(t-1)}u(t-1)\right\}.$$

Computing the inverse Laplace transform finally gives the solution,

$$f(t) = \left(e^{-(t-1)} - e^{-2(t-1)}\right)u(t-1).$$