Analysis III for D-BAUG, Fall 2018 — Lecture 11

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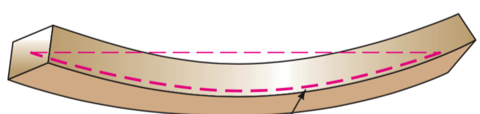
1 Beams (*Balken*)

Beams are basic mechanical systems that appear over and over in civil engineering. We consider a beam of length L whose axis of symmetry is aligned with the x-axis, the left endpoint of the beam is at the origin, and the right endpoint at $x = L$.

Forces acting on the beam cause the beam to deflect. The shape of the beam under the influence of external forces is described by the **deflection curve** $y(x)$.



Here upward is considered the positive direction. Therefore $y(x) > 0$ if the deflection curve lies above the symmetry axis at point x, and $y(x) < 0$ if it lies below. Later we will also consider vertical beams, or **beam columns**.

When we analyze beams, there are two basic cases to consider. The first is the **static** case, where the beam is in static equilibrium and does not move. In this case the deflection curve $y = y(x)$ does not depend on time, and its shape is determined by an ODE. The second case is the **dynamic** case, where there is a net force acting on the beam. This leads to acceleration and implies that the deflection curve changes over time. In this case its shape $y = y(x, t)$ is determined by a PDE. Today we focus on the static case.

*These notes were originally written by Menny Akka and edited by Martin Larsson. Some material was also taken from Alessandra Iozzi's notes.

2 The Euler-Bernoulli beam equation

The deflection curve $y(x)$ of a static beam satisfies the **Euler-Bernoulli beam equation**,

$$EIy''''(x) = f(x),$$

where E is the Young modulus of elasticity, I is the moment of inertia of a cross-section of the beam, and $f(x)$ is the load, or force per unit length, acting on the beam. We will not derive this equation here; let us only mention that the parameter E depends on the material the beam is made of, while I depends on the geometry of the beam.

To solve this fourth-order ODE we need four boundary conditions. What they are depends on how the beam is attached at the endpoints. There are three main types:

(i) Embedded: $y(0) = y'(0) = 0$,

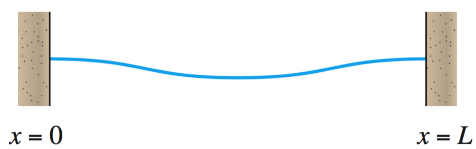
(ii) Free end: $y''(0) = y'''(0) = 0$,

(iii) Simply supported: $y(0) = y''(0) = 0$.

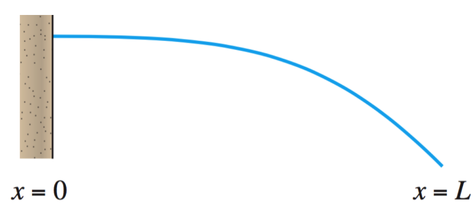
(In all three cases, "0" can be replaced by "L" to describe the other end.)

Example 2.1. Here are a few examples of beams attached in various ways, along with corresponding boundary conditions:

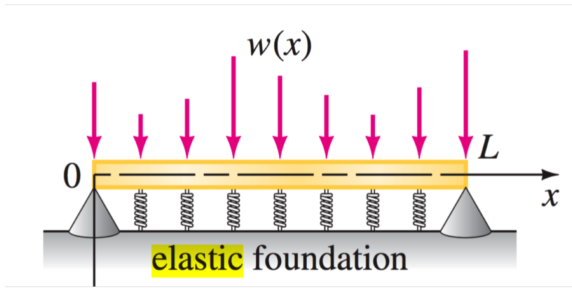
- Embedded at both ends: $y(0) = y'(0) = y(L) = y'(L) = 0$.



- Cantilever beam, for instance a diving board or an airplane wing: $y(0) = y'(0) = y''(L) = y'''(L) = 0$.



- Simply supported at both ends: $y(0) = y''(0) = y(L) = y''(L) = 0$.



3 Elastic foundation

In some situations the beam rests on a foundation which is elastic and contributes an upward force acting along the length of the beam:

The force exerted by the elastic foundation depends on how compressed the foundation is. It is proportional to the deflection of the beam at each point. Adjusting the Euler-Bernoulli equation to account for this results in the ODE

$$EIy''''(x) = f(x) - ky(x),$$

where $k > 0$ is the spring constant (or modulus) associated with the elastic foundation. Note that a negative deflection $y(x) < 0$ leads to the additional positive load $-ky(x)$ on the right-hand side, which is consistent with our convention that the positive direction is upwards. For the purpose of solving this equation it is often convenient to write it as

$$y''''(x) + 4a^4 y(x) = \frac{f(x)}{EI},$$

where $a = \left(\frac{k}{4EI}\right)^{1/4}$.

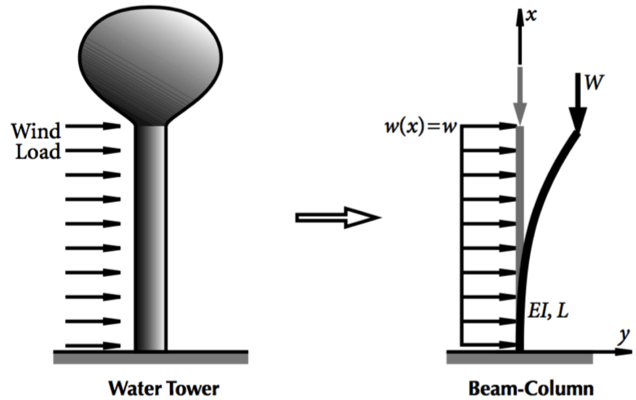
4 Beam columns ("water towers")

Another important type of beam is the beam column, which consists of a vertical beam with a mass at the top:

The deflection curve is now parameterized by x, which runs from 0 (the bottom of the beams) to L (the top of the beam), and we choose rightwards[1] as the positive direction for the deflection curve $y(x)$. Adjusting the Euler-Bernoulli equation to account for this situation results in the ODE

$$y''''(x) + \frac{W}{EI} y''(x) = \frac{f(x)}{EI}.$$

[1]Note that the convention is changed form the first version of this lecture note in order to match the picture. This is just a convention and does not effect the way the exercises are solved.



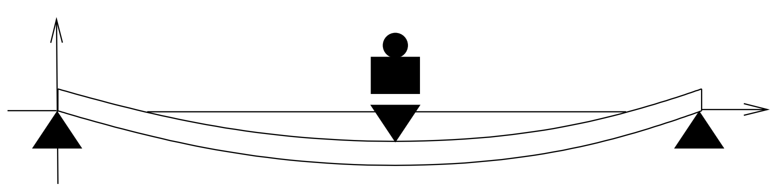
Here $f(x)$ is an external horizontal load on the beam, for example the wind pressure $w(x) = w$ in the picture. Again, $f(x)$ is positive if the load acts rightwards. The mass W (the water on top) is called the **axial load**.

5 Solving the beam equation using the Laplace transform

The Laplace transform has two key advantages that makes it a very useful tool for solving the beam equation: (i) differential equations become algebraic equations, and (ii) the Laplace transform interacts nicely with the Heaviside and Dirac delta functions.

Small mathematical issue: We'll be computing the Laplace transform $Y(s) = \mathcal{L}\{y(x)\} = \int_0^\infty e^{-sx} y(x)dx$ of the deflection curve $y(x)$. However, $y(x)$ is only defined for $0 \leq x \leq L$, so how can we perform this calculation? The answer is simple: we view $y(x)$ as a function defined for all $x \geq 0$, take the Laplace transform, solve for $Y(s)$, and compute the inverse Laplace transform to obtain $y(x)$ (for all $x \geq 0$). However, only the values of $y(x)$ for $0 \leq x \leq L$ matter for the solution of the beam equation.

We'll solve the three-point beam bending problem:



That is, we have a simply supported beam with a downward point load of size F at location $x = a$. Notice how the Dirac delta function allows us to capture point loads: we want our downward load $f(x)$ be zero for $x \neq a$, and at the same time have a total load $\int_0^L f(x)dx = F$. This is exactly what the Dirac delta function accomplishes via the sifting property.

This leads to the following problem:

Find $y = y(x)$ such that

$$(1) \qquad \begin{cases} EIy''''(x) = -F\delta(x-a), \quad 0 < x < L, & \text{(ODE)} \\ y(0) = y''(0) = 0 & \\ y(L) = y''(L) = 0 & \text{(BC)} \end{cases}$$

Example 5.1. Let's solve the problem (1) with $EI = 1$, $L = 2$, $F = 1$, $a = 1$. The ODE then becomes

$$y''''(x) = -\delta(x-1).$$

We could integrate this directly, but let's instead use the Laplace transform to see how it works. With $Y(s) = \mathcal{L}\{y(x)\}$ we get

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = -e^{-s}.$$

Here $y(0) = y''(0) = 0$ due to the boundary conditions. As we do not yet know the values of the other derivatives at zero, let's write $c_1 = y'(0)$ and $c_2 = y'''(0)$ for the moment. Then

$$Y(s) = \frac{-e^{-s}}{s^4} + \frac{c_1}{s^2} + \frac{c_2}{s^4}.$$

From the table of Laplace transforms we know that $\mathcal{L}^{-1}\{\frac{1}{s^2}\} = x$ and $\mathcal{L}^{-1}\{\frac{1}{s^4}\} = \frac{x^3}{6}$. By the t-shifting property, $\mathcal{L}^{-1}\{\frac{1}{s^4}e^{-s}\} = \frac{(x-1)^3}{6}u(x-1)$, where $u(x)$ is the Heaviside function. Therefore,

$$y(x) = \frac{-(x-1)^3}{6}u(x-1) + c_1x + \frac{c_2}{6}x^3. \tag{5.1}$$

It remains to identify c_1 and c_2. For this we use the remaining boundary conditions. First, we have

$$0 = y''(2) = \frac{-3 \times 2 \times (2-1)}{6} + 0 + \frac{c_2}{6} \times 3 \times 2 \times 2 = -1 + 2c_2$$

and therefore $c_2 = 1/2$. Then we get

$$0 = y(2) = \frac{-(2-1)^3}{6} + c_1 \times 2 + \frac{1}{2 \times 6} \times 2^3 = \frac{1}{2} + 2c_1,$$

and therefore $c_1 = -1/4$. Plugging these values of c_1 and c_2 into (5.1) we finally end up with the solution,

$$y(x) = \frac{-(x-1)^3}{6}u(x-1) - \frac{x}{4} + \frac{x^3}{12}.$$

Example 5.2. You may be asked to translate a diagram of the following type into a corresponding beam equation:

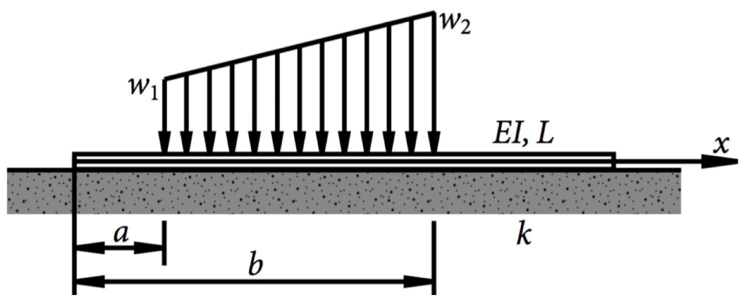


Figure 1: A beam with free ends and foundation with modulus k.

In this case, we clearly need to use the beam equation with elastic foundation. That is, we should consider the ODE

$$EIy''''(x) + ky(x) = f(x).$$

Since the beam has free ends, the boundary conditions are

$$y''(0) = y'''(0) = y''(L) = y'''(L) = 0.$$

Finally, we need to find the function $f(x)$. It is supposed to be zero outside the interval $[a, b]$, and move linearly from the value $-w_1$ at $x = a$ to the value $-w_2$ at $x = b$ (pay attention to the minus signs: the load is directed downwards!). This can be done by first finding the linear function through $-w_1$ at a and $-w_2$ at b, namely

$$-\frac{x-a}{b-a}w_2 - \frac{x-b}{a-b}w_1.$$

This function is not zero outside $[a, b]$, but we can use the Heaviside function to make it zero there: simply multiply by $u(x - a) - u(x - b)$. Indeed, this factor is one for $a \leq x < b$ and zero otherwise. Therefore we get

$$f(x) = -\left(\frac{x-a}{b-a}w_2 + \frac{x-b}{a-b}w_1\right)\left(u(x-a) - u(x-b)\right).$$

Now you can solve the resulting equation using the Laplace transform!