Analysis III for D-BAUG, Fall 2018 — Lecture 2

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0 Recall from last time:

Last time we considered (among other things) the classification of **second-order linear PDE** in two variables, namely PDE of the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$$

where the coefficients A, B, C, D, E, F, G are either constant, or functions of the independent variables x, y. The three key representative examples of such PDE are (here α is some given constant):

1-dimensional heat equation: $u_t = \alpha^2 u_{xx}$ (parabolic)

1-dimensional wave equation: $u_{tt} = \alpha^2 u_{xx}$ (hyperbolic)

2-dimensional Laplace equation: $u_{xx} + u_{yy} = 0$ (elliptic)

The next few lectures will be about the 1-dimensional heat equation.

1 The heat equation: Initial-Boundary Value Problem (1)

The following equation is a model for heat flow in a laterally insulated thin rod:

Find $u = u(x,t)$ such that

$$\text{(IBVP)} \quad \begin{cases} u_t = \alpha^2 u_{xx} & \text{in } \Omega = (0,L) \times (0,\infty), & \text{(PDE)} \\ u(0,t) = u(L,t) = 0 & \text{for all } t > 0, & \text{(BC)} \\ u(x,0) = \phi(x) & \text{for all } x \in (0,L). & \text{(IC)} \end{cases}$$

Here α and $L > 0$ are given constants, and $\phi(x)$ is a given function.

This is called an **Initial-Boundary Value Problem (IBVP)**. This is because it asks for a function $u(x,t)$ which not only satisfies the PDE in the region Ω, but also has a prescribed behavior on the boundary of this region, as well as at the initial time $t = 0$. Indeed, it is required that

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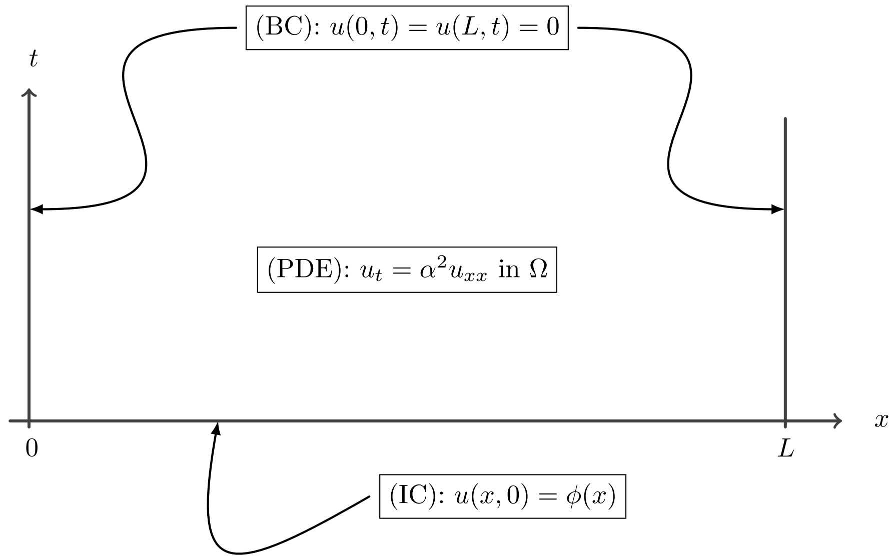
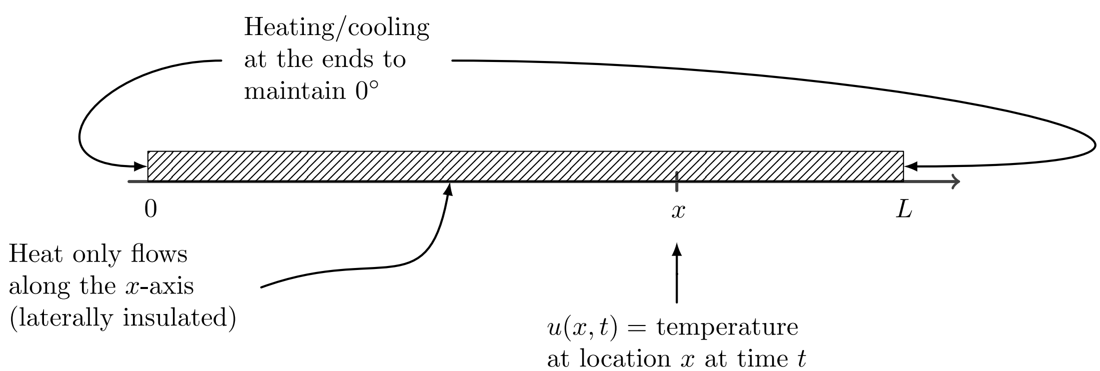
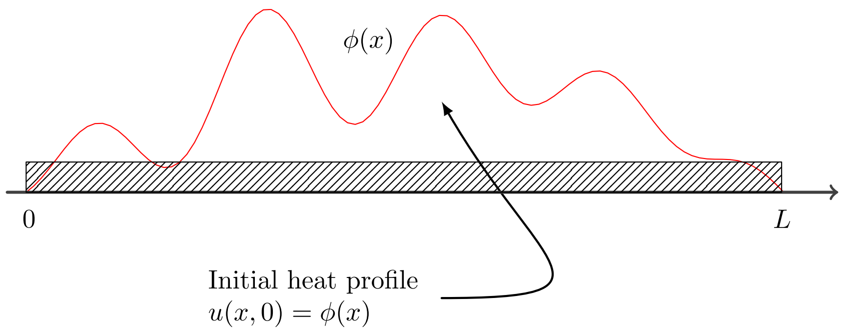
$u(0,t) = u(L,t) = 0$ for all $t > 0$ (the **boundary condition (BC)**), and that $u(x,0) = \phi(x)$ for all $x \in (0,L)$ (the **initial condition (IC)**). One can visualize this by drawing the following space-time picture:



The IBVP above is a model for heat flow in a laterally insulated thin rod of length L. More specifically, $u(x,t)$ represents the temperature at location x and time t. The boundary conditions (BC) state that the endpoints of the rod are kept at zero degrees at all times. The initial condition (IC) states that the initial heat distribution at time $t = 0$ is given by the function $\phi(x)$. This is illustrated in the following two figures:



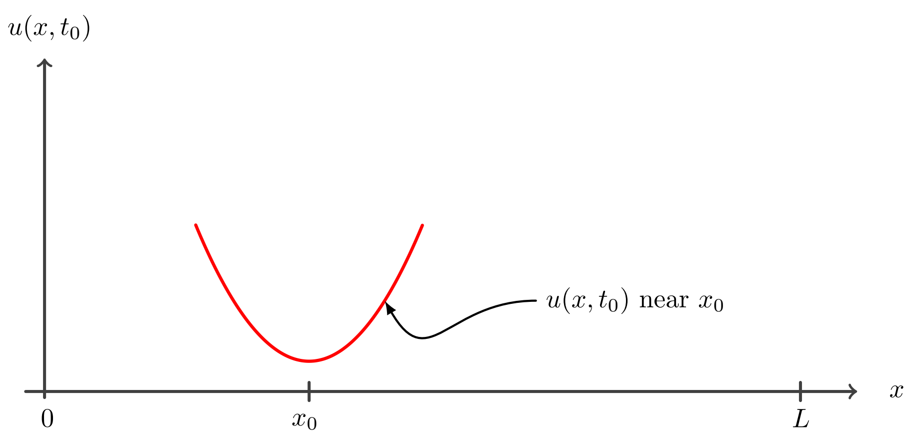


While the meaning of (BC) and (IC) is fairly clear, it is much less obvious why the PDE itself, $u_t = \alpha^2 u_{xx}$, is a reasonable model for heat flow. We will not give a full derivation here, but only a brief explanation of why this PDE is consistent with our intuition of how heat flow behaves. The key physical principle is the following:

The temperature at any given point tends to move
toward the average temperature at nearby points.

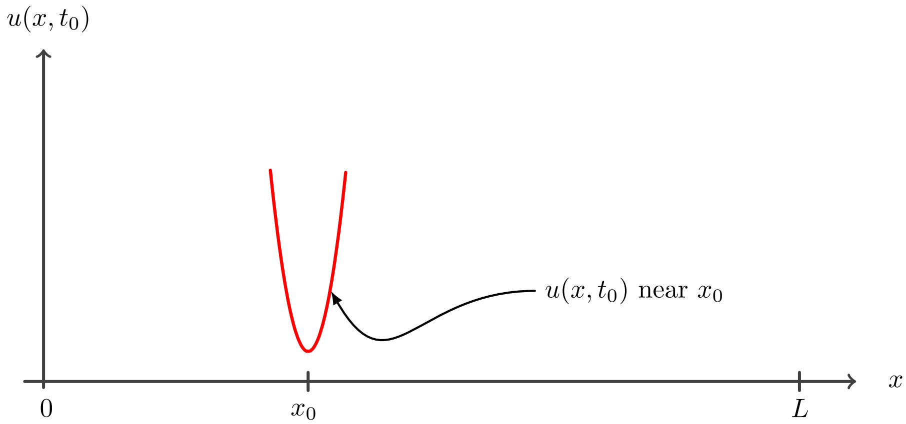
Let's see how the PDE is consistent with this principle. Consider a location $x_0 \in (0, L)$ somewhere along the rod, and a time point $t_0 > 0$. Then $u(x_0, t_0)$ is the temperature at this time and location. Let's think about how $u_t(x_0, t_0)$, which describes the *rate of change in temperature at location* x_0, ought to depend on the temperature at nearby points.

Suppose $u_{xx}(x_0, t_0) > 0$. Then $u(x, t_0)$ viewed as a function of x is convex near x_0:



Therefore, the average temperature near x_0 is larger than the temperature $u(x_0, t_0)$ at x_0. But then the temperature at x_0 should be increasing, meaning that $u_t(x_0, t_0) > 0$. This is consistent with the PDE.

Now, if $u_{xx}(x_0, t_0)$ is not only positive, but very large, the convexity of the function $u(x, t_0)$ around x_0 is more pronounced:



The average temperature near x_0 is now significantly larger than the temperature at x_0. Thus the temperature at x_0 should be rising rather quickly, meaning that $u_t(x_0, t_0)$ should be rather large. This is again consistent with the PDE, which states that u_t is actually *proportional* to u_{xx}, and is therefore large whenever u_{xx} is.

An analogous discussion (with convexity replaced by concavity) can be carried out when $u_{xx}(x_0, t_0) < 0$. The conclusion is again that the PDE is consistent with the principle that the temperature at a point tends to move toward the average temperature at nearby points.

Remark 1.1. In the IBVP above, we could certainly use a different constant than zero for the boundary conditions. This would not affect the solution method. Indeed, if (BC) is replaced by $u(0, t) = u(L, t) = c$ for some constant c, then a function $v(x, t)$ solves this modified IBVP if and only if the function $u(x, t) = v(x, t) - c$ solves the IBVP above (with zero boundary conditions, but $\phi(x)$ replaced by $\phi(x) - c$ in the initial condition). *Exercise: Check this!*

What happens if the extremities are kept at different temperatures instead?

2 Solving the IBVP via separation of variables

We are now going to solve the IBVP in three steps, for certain functions $\phi(x)$ in (IC). First, we look for a family of functions $u(x, t)$ that satisfy the PDE in Ω. Second, we single out those functions that in addition satisfy (BC). Third, we find a family of specific initial conditions $\phi(x)$ so that we can solve the corresponding IBVP using the functions $u(x, t)$ that we found. As it turns out, these cases will be the stepping stone towards solving the IBVP in the general case in future lectures.

We make the following *separation of variables Ansatz*:

$$u(x, t) = X(x)T(t), \tag{2.1}$$

where $X(x)$ and $T(t)$ are unknown functions.

Step I: Find the general form of $X(x)$ and $T(t)$

Plugging the Ansatz (2.1) into the PDE leads to the equation

$$X(x)T'(t) = \alpha^2 X''(x)T(t).$$

Re-writing this equation yields

$$\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)}.$$

But since this holds for all x and t, both sides must be equal to some constant K. (To see this, plug in for example $t = 3$ to get $X''(x)/X(x) = K$ for all x, where $K = (1/\alpha^2)T'(3)/T(3)$.) This leads to the two ODEs

$$\begin{aligned} X''(x) &= KX(x), \\ T'(t) &= \alpha^2 KT(t). \end{aligned} \tag{2.2}$$

Our one original PDE has been replaced by two ODEs! This is nice, because we know how to solve such ODEs. In particular, the equation for $T(t)$ has the solution

$$T(t) = Ce^{\alpha^2 Kt}$$

for some constant C.

The discussion now splits into 3 cases depending on the sign of K. The first two cases, as we will see in Step II, only lead to the zero solution, so the only interesting one is the third one.

Case $K = 0$. In this case $T(t) = C$. Also, $X''(x) = 0$ for all x, so that $X(x)$ is a linear function, that is to say $X(x) = \hat{A}x + \hat{B}$ for some constants $\hat{A}, \hat{B}$. Hence, $u(x,t) = X(x)T(t) = Ax + B$, where $A = \hat{A}C$, $B = \hat{B}C$.

Case $K > 0$. This can also be written $K = \lambda^2$ for some real number $\lambda > 0$. The first ODE then becomes $X''(x) = \lambda^2 X(x)$. The solution, as you know from previous courses, is $X''(x) = \hat{A}e^{\lambda x} + \hat{B}e^{-\lambda x}$ for some constants $\hat{A}, \hat{B}$. Hence, $u(x,t) = e^{\alpha^2\lambda^2 t}(Ae^{\lambda x} + Be^{-\lambda x})$, where $A = \hat{A}C$, $B = \hat{B}C$.

Case $K < 0$. This can also be written $K = -\lambda^2$ for some real number $\lambda > 0$. The first ODE in (2.2) becomes $X''(x) = -\lambda^2 X(x)$. The solution is $X(x) = \hat{A}\sin(\lambda x) + \hat{B}\cos(\lambda x)$, where $\hat{A}$, $\hat{B}$ are arbitrary constants. Multiplying $X(x)$ and $T(t)$, and setting $A = \hat{A}C$, $B = \hat{B}C$, we obtain:

$$u(x,t) = e^{-\alpha^2\lambda^2 t}\left(A\sin(\lambda x) + B\cos(\lambda x)\right) \tag{2.3}$$

for some constants A, B, and $\lambda > 0$.

Step II: Match (BC)

We now determine those constants A, B, and λ such that (2.3) satisfies (BC), that is to say $u(0,t) = u(L,t) = 0$ for all t.

Case $K = 0$. In this case from $u(0,t) = 0$ we get $B = 0$, and then from $u(L,t) = 0$ we get $A = 0$. So $u(x,t) = 0$.

Case $K > 0$. Since $T(t) = e^{\alpha^2\lambda^2 t}$ is always non-zero, from $u(0,t) = 0$ we get $A + B = 0$, and hence from $u(L,t) = 0$ we get $A(e^{\lambda L} - e^{-\lambda L}) = 0$. But then $A = 0$, and $B = -A = 0$. So, again, $u(x,t) = 0$.

Case $K < 0$. We finally get to the case leading to interesting solutions. First, consider the condition $u(0,t) = 0$. Since $\sin(0) = 0$ and $\cos(0) = 1$, this condition states that

$$e^{-\alpha^2\lambda^2 t}B = u(0,t) = 0.$$

Thus $B = 0$. Next, consider the condition $u(L,t) = 0$. Since $B = 0$, we this condition states that

$$e^{-\alpha^2\lambda^2 t} A\sin(\lambda L) = u(L,t) = 0.$$

For $A = 0$ we just get the zero solution, so that we can assume that this is not the case. Therefore we must have $\sin(\lambda L) = 0$. This happens precisely when

$$\lambda = \frac{n\pi}{L} \quad \text{for some } n \in \mathbb{Z}.$$

We conclude:

> Any solution of the form (2.1) that also satisfies (BC) is given by $u(x,t) = Au_n(x,t)$ for some constant A and some $n \in \mathbb{Z}$, where $u_n(x,t)$ denotes the ***n*th sine building block**, given by
>
> $$u_n(x,t) = e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right). \tag{2.4}$$

(The only solution that came out of the other two cases, $u(x,t) = 0$, corresponds to $A = 0$, so it is included.)

Remark 2.1. Because of the factor $e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$, we have that $u_n(x,t) \to 0$ as $t \to +\infty$, for any x. This says that the temperature along the whole rod tends to 0, which is of course what happens to a real-world rod in the situation we are modeling.

Step III: Match (IC)

We aren't yet able to match the initial condition (IC) for arbitrary functions $\phi(x)$. Indeed, looking at the conclusion from Step II, we see that we have no more freedom apart from the choice of constant A and integer n. Nonetheless, the nth sine building block $u_n(x,t)$ does satisfy *some* IBVP, namely the following:

$$(\text{IBVP})_n \quad \begin{cases} u_t = \alpha^2 u_{xx} & \text{in } \Omega = (0,L) \times (0,\infty), \\ u(0,t) = u(L,t) = 0 & \text{for all } t > 0, \\ u(x,0) = \sin\left(\frac{n\pi}{L}x\right) & \text{for all } x \in (0,L). \end{cases}$$

Notice what happened here: we hand-picked $\phi(x)$ to be equal to $u_n(x,0)$! This trick might look puzzling, but will turn out to be useful. At least it is clear that $u_n(x,t)$ satisfies this IBVP.

Now we will use the superposition principle to construct solutions of the original IBVP for a much larger class of initial conditions. The main ingredient is the superposition principle. The following is true, where a and b are arbitrary constants:

- If v and w are solutions of (PDE), then so is $u = av + bw$.
- If v and w both satisfy (BC), then so does $u = av + bw$.
- If $v(x,0) = \phi(x)$ and $w(x,0) = \psi(x)$, then the function $u = av + bw$ satisfies $u(x,0) = a\phi(x) + b\psi(x)$.

Here the first point is just a special case of the superposition principle from Lecture 1 (note that our PDE is homogeneous!). The second and third points are obvious.

Using these three points together with the fact that $u_n(x,t)$ is the solution of $(\text{IBVP})_n$, we see that for any constants a_1 and a_2, the function

$$a_1 u_1(x,t) + a_2 u_2(x,t)$$

is the solution of (IBVP) with $\phi(x) = a_1 \sin\left(\frac{\pi x}{L}\right) + a_2 \sin\left(\frac{2\pi x}{L}\right)$. Repeating this any number of times, the following general formula is obtained:

Let α, $L > 0$, $a_1, \ldots, a_N$ be constants. The solution of the IBVP

$$\begin{cases} u_t = \alpha^2 u_{xx} & \text{in } \Omega = (0,L) \times (0,\infty), & \text{(PDE)} \\ u(0,t) = u(L,t) = 0 & \text{for all } t > 0, & \text{(BC)} \\ u(x,0) = \sum_{n=1}^{N} a_n \sin\left(\frac{n\pi x}{L}\right) & \text{for all } x \in (0,L) & \text{(IC)} \end{cases}$$

is given by

$$u(x,t) = \sum_{n=1}^{N} a_n u_n(x,t) = \sum_{n=1}^{N} a_n e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right)$$

This solution is somewhat limited since it only works for initial conditions of a very specific form: it has to be a linear combination of sine functions. However, we will see later that with some work, this leads to a much more general class of initial conditions.