Analysis III for D-BAUG, Fall 2018 — Lecture 3

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0 Recall from last time:

Last time we considered the IBVP (initial-boundary value problem) that models heat flow on a laterally insulated thin rod with temperature zero at the endpoints. We will now continue with two other IBVPs, corresponding to (i) heat flow on a completely insulated rod, and (ii) heat flow on a completely insulated circular wire. Just as last time, at first we will not be able to solve these IBVPs for any arbitrary initial condition. However, toward the end of this lecture, we will begin to look at the theory that will help us fix this deficiency: the theory of Fourier series.

1 The heat equation: IBVP (2)

The following equation is a model for heat flow in a completely insulated thin rod:

Find $u = u(x,t)$ such that

$$\begin{cases} u_t = \alpha^2 u_{xx} & \text{in } \Omega = (0,L) \times (0,\infty), & \text{(PDE)} \\ u_x(0,t) = u_x(L,t) = 0 & \text{for all } t > 0, & \text{(BC)} \\ u(x,0) = \phi(x) & \text{for all } x \in (0,L). & \text{(IC)} \end{cases}$$

Here α and $L > 0$ are given constants, and $\phi(x)$ is a given function.

Notice the difference compared to the IBVP from last time: now (BC) involves the derivative $u_x(x,t)$ rather than the function value itself. The (BC) captures the idea that the endpoints are completely insulated. Indeed, it is known in physics that heat flow along the rod is proportional to the derivative $u_x(x,t)$. Intuitively this makes sense, because is, say, $u_x(x_0,t_0) > 0$ at time t_0 and location x_0 along the rod, then the temperature is higher immediately to the right of x_0 than to the left of x_0. Therefore heat should flow from right to left. Insulating the rod at the endpoints means that no heat can flow in or out. This is captured by the zero derivative requirement $u_x(0,t) = u_x(L,t) = 0$.

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2 Solving IBVP (2)

The procedure is very similar to last time, so we will skip details. Step I is actually identical; here is a review. We consider the Ansatz

$$u(x,t) = X(x)T(t) \tag{2.1}$$

and plug this into the PDE. We obtain, for some constant K, the two ODEs

$$X''(x) = KX(x), \qquad T'(t) = \alpha^2 K T(t).$$

Subdividing in the 3 cases $K = 0, K > 0$ and $K < 0$ one obtains solutions of the form, respectively,

$$u(x,t) = Ax + B,$$
$$u(x,t) = e^{\alpha^2\lambda^2 t}\left(Ae^{\lambda x} + Be^{-\lambda x}\right),$$
$$u(x,t) = e^{-\alpha^2\lambda^2 t}\left(A\sin(\lambda x) + B\cos(\lambda x)\right),$$

for some constants A, B, and $\lambda > 0$.

Next, in **Step II**, we match the boundary conditions (BC).

The first case only yields the solution $u(x,t) = C$ for some constant C. Notice that this is a solution of the IBVP we are considering now, but not the one from the last lecture unless $C = 0$.

Let us analyse the second case. First of all, $u_x(x,t) = e^{\alpha^2\lambda^2 t}\left(A\lambda e^{\lambda x} - B\lambda e^{-\lambda x}\right)$. Setting $u_x(0,t) = 0$ we get $e^{\alpha^2\lambda^2 t}(A\lambda - B\lambda) = 0$. Since $\lambda > 0$, we must have $A = B$. But then from $u_x(L,t) = 0$ we get $e^{\alpha^2\lambda^2 t}\lambda A(e^{\lambda L} - e^{-\lambda L})$, so that $A = 0$. Hence, from this case we only get the solution $u(x,t) = 0$.

Let us now consider the final case, which once again is one yielding the most interesting solutions. Computing the x-derivative of $u(x,t)$ yields

$$u_x(x,t) = e^{-\alpha^2\lambda^2 t}\left(A\lambda\cos(\lambda x) - B\lambda\sin(\lambda x)\right).$$

The first part of (BC), $u_x(0,t) = 0$, gives $A\lambda = 0$. Since $\lambda > 0$, we have $A = 0$. The second part of (BC), $u_x(L,t) = 0$, gives $B\lambda\sin(\lambda L) = 0$. For $B = 0$, we just get $u(x,t) = 0$. For $\sin(\lambda L) = 0$, we get

$$\lambda = \frac{n\pi}{L} \quad \text{for some } n \in \mathbb{Z}.$$

We were requiring $\lambda > 0$, which would force $n > 0$. If we also allow $n = 0$, we get $\lambda = 0$ and recover the solution $u(x,t) = C$ from the first case. We deduce that constant multiples of the functions

$$e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\cos\left(\frac{n\pi}{L}x\right), \qquad n \in \mathbb{Z}, \tag{2.2}$$

cover all solutions of the form (2.1).

Finally, in **Step III**, we try to match the initial condition (IC). Again, at this point we can only deal with some particular choices of $\phi(x)$. We observe that the functions in (4.3) solve the following IBVPs:

$$(\text{IBVP})_n \quad \begin{cases} u_t = \alpha^2 u_{xx} & \text{in } \Omega = (0,L)\times(0,\infty), \\ u_x(0,t) = u_x(L,t) = 0 & \text{for all } t > 0, \\ u(x,0) = \cos\left(\frac{n\pi}{L}x\right) & \text{for all } x \in (0,L). \end{cases}$$

To continue, we once again rely on the superposition principle: If v (respectively w) solves the original IBVP with initial condition $\phi(x)$ (respectively $\psi(x)$), and if a and b are constants, then the function $u = av + bw$ solves the same IBVP, but now with initial condition $a\phi(x) + b\psi(x)$. Repeating this gives us the following general formula:

Let α, $L > 0$, $a_0, \ldots, a_N$ be constants. The solution of the IBVP

$$\begin{cases} u_t = \alpha^2 u_{xx} & \text{in } \Omega = (0, L) \times (0, \infty), & \text{(PDE)} \\ u_x(0, t) = u_x(L, t) = 0 & \text{for all } t > 0, & \text{(BC)} \\ u(x, 0) = a_0 + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) & \text{for all } x \in (0, L) & \text{(IC)} \end{cases}$$

is given by

$$u(x, t) = a_0 + \sum_{n=1}^{N} a_n e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi}{L} x\right)$$

Remark 2.1. Notice the constant a_0 in the general formula! This comes from taking $n = 0$ (corresponding to $\lambda = 0$) in (4.3). The remaining terms tend to 0 as $t \to \infty$, so that the equation predicts that the temperature becomes (almost) constant along the rod after a sufficiently long time. *Exercise:* Show that a_0 is in fact the average temperature at time 0, that is to say $a_0 = \frac{1}{L} \int_0^L \phi(x) dx$.

3 The heat equation: IBVP (3)

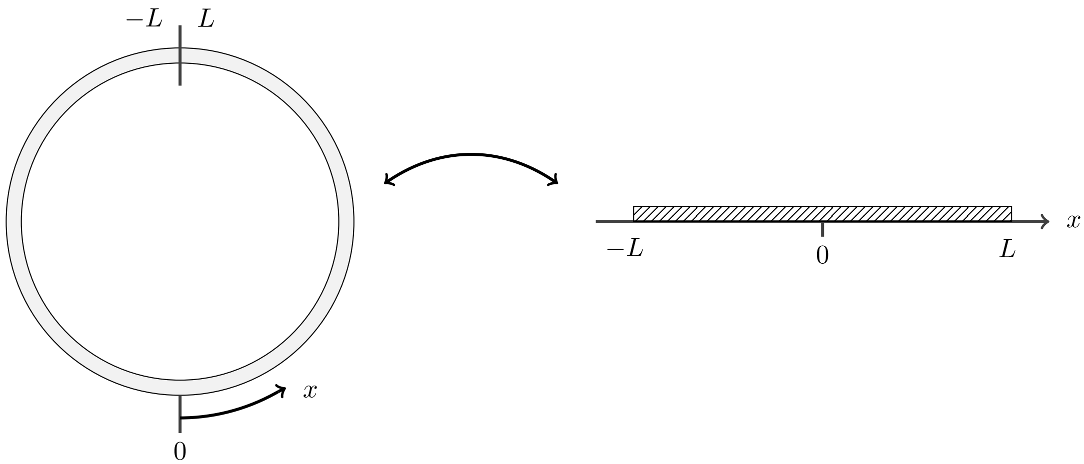
The following equation is a model for heat flow in an insulated circular wire:

Find $u = u(x, t)$ such that

$$\begin{cases} u_t = \alpha^2 u_{xx} & \text{in } \Omega = (-L, L) \times (0, \infty), & \text{(PDE)} \\ u(-L, t) = u(L, t) & & \\ u_x(-L, t) = u_x(L, t) & \text{for all } t > 0, & \text{(BC)} \\ u(x, 0) = \phi(x) & \text{for all } x \in (-L, L). & \text{(IC)} \end{cases}$$

Here α and $L > 0$ are given constants, and $\phi(x)$ is a given function.

Notice the difference compared to the previous IBVPs: Now x ranges over the interval $[-L, L]$ rather than $[0, L]$. Also, (BC) now involves conditions both on the function itself and its x-derivative. This IBVP models heat flow on an insulated circular wire of length $2L$:



The boundary condition (BC) is very easy to understand. Indeed, x signifies the location along the wire, and since the wire forms a circular loop, the two values $x = L$ and $x = -L$ actually correspond to the same point on the wire. With this in mind it is obvious that we must have $u(-L, t) = u(L, t)$ and $u_x(-L, t) = u_x(L, t)$.

4 Solving IBVP (3)

The procedure is again very similar, so we will skip details. We start from the Ansatz

$$u(x, t) = X(x)T(t). \tag{4.1}$$

Step I is identical. We get solutions of 3 possible forms. We will skip two of the cases, which only yield the solution $u(x, t) = C$ (*exercise*), and we focus on solutions of the form

$$u(x, t) = e^{-\alpha^2\lambda^2 t}\left(A\sin(\lambda x) + B\cos(\lambda x)\right) \tag{4.2}$$

for some constants A, B, and $\lambda > 0$. Matching the boundary conditions (BC) in **Step II** yields the equations

$$A\sin(-\lambda L) + B\cos(-\lambda L) = A\sin(\lambda L) + B\cos(\lambda L),$$
$$A\lambda\cos(-\lambda L) - B\lambda\sin(-\lambda L) = A\lambda\cos(\lambda L) - B\lambda\sin(\lambda L).$$

Thanks to the identities $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$, these equations hold if and only if

$$2A\sin(\lambda L) = 0 \qquad \text{and} \qquad 2B\lambda\sin(\lambda L) = 0.$$

If $A = B = 0$ we just get the zero solution. If not, we get

$$\lambda = \frac{n\pi}{L} \quad \text{for some } n \in \mathbb{Z},$$

just as before (with the same note about the case $n = 0$). We deduce in particular that all the functions

$$e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\cos\left(\frac{n\pi}{L}x\right) \qquad \text{and} \qquad e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi}{L}x\right), \qquad n \in \mathbb{Z}, \tag{4.3}$$

are solutions of the form (2.1). Finally, in **Step III**, we use the superposition principle to obtain the following general formula:

Let α, $L > 0$, $a_0, \ldots, a_N$, $b_1, \ldots, b_N$ be constants. The solution of the IBVP

$$\begin{cases} u_t = \alpha^2 u_{xx} & \text{in } \Omega = (-L, L) \times (0, \infty), \quad \text{(PDE)} \\ u(-L,t) = u(L,t) & \\ u_x(-L,t) = u_x(L,t) & \text{for all } t > 0, \quad \text{(BC)} \\ u(x,0) = \phi(x) & \text{for all } x \in (-L, L), \quad \text{(IC)} \end{cases}$$

where $\phi(x) = a_0 + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)$, is given by

$$u(x,t) = a_0 + \sum_{n=1}^{N} e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)$$

Remark 4.1. Notice once again that as t tending to infinity $u(x,t)$ tends to a_0. Once again, it turns out that a_0 is the average temperature at time 0, that is to say $a_0 = \frac{1}{2L}\int_{-L}^{L} \phi(x)dx$.

5 General (IC): Fourier series

So far, we have been able to solve various IBVPs for initial conditions $\phi(x)$ of the form

$$\phi(x) = a_0 + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right).$$

Such functions are called **trigonometric polynomials**. Of course, lots of functions $\phi(x)$ that we might be interested in are not of this form. For example, common functions like $\phi(x) = x^k$ or $\phi(x) = e^{\beta x}$ are excluded. Nonetheless, we did put in a lot of work, and we'd like to see how far it can take us. The idea is to let $N \to \infty$, and consider initial conditions which are **trigonometric series**:

$$\phi(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right). \tag{5.1}$$

(Notice that the sum now contains infinitely many terms!) As long as there is an infinite version of the superposition principle, we ought to be able to solve the IBVPs corresponding to such initial conditions. This leads to the following question, which was answered by Joseph Fourier:

Which functions $\phi(x)$ can be expressed as trigonometric series? Furthermore, given such a function, how does one compute the coefficients a_n, b_n?