Analysis III for D-BAUG, Fall 2018 — Lecture 4

Lecturer: Alex Sisto (sisto@math.ethz.ch)*

0 Recall from last time:

So far we know how to solve our IBVPs with initial conditions $\phi(x)$ which are trigonometric polynomials of the form

$$\phi(x) = \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi}{L}x\right) \qquad \text{(for IBVP (1))}$$

$$\phi(x) = a_0 + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi}{L}x\right) \qquad \text{(for IBVP (2))}$$

$$\phi(x) = a_0 + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right) \qquad \text{(for IBVP (3))}$$

By taking $N = \infty$, we can even solve IBVPs whose initial conditions consist of the corresponding trigonometric series.[1] Last time we asked the question which functions admit such representations, and how one can recover the coefficients a_n, b_n. In this lecture we will provide the answer.

1 Fourier theorem

The answer to the first question, which functions can be represented by a trigonometric series, is simple, very pleasing, and almost a miracle. The following is a beautiful theorem of Fourier.

*These notes were originally written by Menny Akka and edited by Martin Larsson. Some material was also taken from Alessandra Iozzi's notes.

[1] To make this mathematically rigorous one must take care when passing to infinite N. We will not dwell on this point here.

Theorem 0. Let $\phi(x)$, $x \in [-L, L]$, be any (differentiable) function. Then there exist coefficients a_n, b_n so that

$$\phi(x) = a_0 + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right)$$

for all $x \in (-L, L)$.

Proving the theorem is beyond the scope of this course, but one does not need to understand why it's true to use it. We now focus on how, given $\phi(x)$, one can determine the corresponding coefficients (the theorem would be no good if we had no idea what the coefficients are after all...).

2 Orthogonality of the trigonometric system

In order to recover the coefficients, we will use the following "orthogonality" relations:

Proposition 2.1 (Orthogonality of the trigonometric system)**.** *Let $m, n \in \mathbb{Z}$ be nonnegative integers. Then:*

$$\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \\ 2L & n = m = 0 \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) dx = 0 \quad \textit{for all } m, n.$$

Proof. Let's focus on the case $L = \pi$ to save ink. The general case uses the same reasoning. To prove the first formula, we use the trigonometric identity

$$\cos(nx)\cos(mx) = \frac{1}{2}\cos\left((n+m)x\right) + \frac{1}{2}\cos\left((n-m)x\right).$$

This gives

$$\begin{aligned} \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\, dx &= \frac{1}{2}\int_{-L}^{L} \cos\left((n+m)x\right) dx + \frac{1}{2}\int_{-L}^{L} \cos\left((n-m)x\right) dx \\ &= \begin{cases} \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0, & n \neq m, \\ \frac{1}{2} \cdot 0 + \frac{1}{2}\int_{-\pi}^{\pi} dx, & n = m \neq 0, \\ \frac{1}{2}\int_{-\pi}^{\pi} dx + \frac{1}{2}\int_{-\pi}^{\pi} dx, & n = m = 0. \end{cases} \end{aligned}$$

The first formula follows. The second and third formulas follow from similar calculations, but now relying instead on the trigonometric identities

$$\begin{aligned}\sin(nx)\sin(mx) &= \frac{1}{2}\cos\left((n-m)x\right) - \frac{1}{2}\cos\left((n+m)x\right), \\ \sin(nx)\cos(mx) &= \frac{1}{2}\sin\left((n+m)x\right) + \frac{1}{2}\sin\left((n-m)x\right).\end{aligned}$$

□

Application: Recover a_n and b_n from $\phi(x)$. Let's put the proposition to use. Suppose we are given a function $\phi(x)$, $x \in [-L, L]$, as a trigonometric series:

$$\phi(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right). \tag{2.1}$$

We'll compute the integral $\int_{-L}^{L}\phi(x)\cos\left(\frac{m\pi}{L}x\right)dx$ for a fixed integer $m \geq 0$. The idea is that this should pick out one of the coefficients of the trigonometric series, essentially just because most of the integrals we computed in the proposition above are 0. This actually works, as we'll see in a minute. Integrating term by term, we obtain

$$\begin{aligned}\int_{-L}^{L}\phi(x)\cos\left(\frac{m\pi}{L}x\right)dx = a_0\int_{-L}^{L}&\cos\left(\frac{m\pi}{L}x\right)dx \\ &+ \sum_{n=1}^{\infty} a_n \int_{-L}^{L}\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx \\ &\quad + \sum_{n=1}^{\infty} b_n \int_{-L}^{L}\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx.\end{aligned}$$

We now apply the proposition. If $m = 0$, then all the integrals on the right-hand side are zero except the first one, which is equal to $2L$. Therefore,

$$\int_{-L}^{L}\phi(x)dx = a_0 2L.$$

If instead $m \geq 1$, then the first integral is zero, as are all the integrals involving both sine and cosine. The integrals involving two cosines are also zero, except one of them, namely the one where $n = m$. That integral is equal to L, and the coefficient in front is $a_n = a_m$. Therefore,

$$\int_{-L}^{L}\phi(x)\cos\left(\frac{m\pi}{L}x\right)dx = a_m L, \qquad m \geq 1.$$

If we compute the integral $\int_{-L}^{L}\phi(x)\sin\left(\frac{m\pi}{L}x\right)dx$ in a similar fashion, we find that it is equal to $b_m L$ for any fixed $m \geq 1$. The result of these computations is the following:

If $\phi(x)$, $x \in [-L, L]$, is given as a trigonometric series (2.1), then

$$a_0 = \frac{1}{2L} \int_{-L}^{L} \phi(x) dx, \tag{2.2}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} \phi(x) \cos\left(\frac{n\pi}{L} x\right) dx, \qquad n \geq 1, \tag{2.3}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} \phi(x) \sin\left(\frac{n\pi}{L} x\right) dx, \qquad n \geq 1. \tag{2.4}$$

The important message here is that the coefficients a_n and b_n can be recovered *using only knowledge of the function values* $\phi(x)$.

The trigonometric series that we obtain is very important for us, so let us give it a name:

Definition 2.2. *Let* $\phi(x)$, $x \in [-L, L]$, *be a given function. Its* **Fourier series** *is the function* $\mathrm{FS}_\phi(x)$ *given by*

$$\mathrm{FS}_\phi(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L} x\right) + b_n \sin\left(\frac{n\pi}{L} x\right)\right), \qquad x \in [-L, L],$$

where the coefficients a_n *and* b_n *are defined by* (2.2)–(2.4)*, and thus only depend on* ϕ.

We can now reformulate Theorem 0:

Theorem 1. Let $\phi(x)$, $x \in [-L, L]$ be any (differentiable) function. Then

$$\mathrm{FS}_\phi(x) = \phi(x) \quad \text{for all } x \in (-L, L).$$

Let's look at an example.

Example 2.3. Take $L = \pi$ and $\phi(x) = x$. Then for any integer $n \geq 0$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx) dx = 0,$$

where we used that x is odd and $\cos(nx)$ is even, therefore $x \cos(nx)$ is odd, and hence its integral over the interval $[-L, L]$ is zero. Alternatively, you can compute the integral using integration by parts as we do now: For any integer $n \geq 1$,

$$\begin{aligned}
b_n = \frac{1}{\pi}\int_{-L}^{L} x\sin(nx)dx &= \frac{1}{\pi}\left(-\frac{x\cos(nx)}{n}\bigg|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos(nx)}{n}dx\right) \\
&= \frac{1}{\pi}\left(-\frac{\pi\cos(n\pi) - (-\pi)\cos(-n\pi)}{n} + \frac{\sin(nx)}{n^2}\bigg|_{-\pi}^{\pi}\right) \\
&= \frac{1}{\pi}\left(-\frac{2\pi\cos(n\pi)}{n} + \frac{\sin(n\pi) - \sin(-n\pi)}{n^2}\right) \\
&= -\frac{2\cos(n\pi)}{n} \\
&= \frac{2}{n}(-1)^{n+1}.
\end{aligned}$$

The theorem says $\phi(x) = \mathrm{FS}_\phi(x)$, or in other words,

$$x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nx).$$

3 The Fourier sine and Fourier cosine series

Using Theorem 1, we can represent functions $\phi(x)$, $x \in [-L, L]$, as linear combinations of sines and cosines. This is precisely what we need to treat initial conditions for IBVP (3). However, for IBVP (1) and IBVP (2), we would like to express the initial condition $\phi(x)$, $x \in [0, L]$, in terms of *only sines* or *only cosines*. As it turns out, this can also be done. We will use a simple trick that involves extending ϕ to a function in $[-L, L]$. We will extend it in a way that, for "symmetry" reasons, the terms of the Fourier series that we do not like disappear.

Lemma 3.1. *If $\phi(x)$, $x \in [-L, L]$, is an* **odd function** *(meaning that $\phi(x) = -\phi(-x)$), then the coefficients* (2.2)–(2.4) *are given by*

$$\begin{aligned}
a_n &= 0, & n \geq 0, \\
b_n &= \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{n\pi}{L}x\right)dx, & n \geq 1.
\end{aligned}$$

If $\phi(x)$, $x \in [-L, L]$, is an **even function** *(meaning that $\phi(x) = \phi(-x)$), then the coefficients* (2.2)–(2.4) *are given by*

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L \phi(x)dx, \\
a_n &= \frac{2}{L}\int_0^L \phi(x)\cos\left(\frac{n\pi}{L}x\right)dx, & n \geq 1, \\
b_n &= 0, & n \geq 1.
\end{aligned}$$

Exercise 3.2. Verify that the lemma is correct.

Definition 3.3. *Consider a function* $\phi(x)$*,* $x \in [0, L]$*. If* $\phi(0) = 0$ *(as is the case for solutions of IBVP (1)), we define its* **odd extension** $\phi^{\text{odd}}(x)$ *by*

$$\phi^{\text{odd}}(x) = \begin{cases} \phi(x), & x \in [0, L], \\ -\phi(-x), & x \in [-L, 0]. \end{cases}$$

If, instead, $\phi'(0) = 0$ *(as is the case for solutions of IBVP (2)), we define its* **even extension** $\phi^{\text{even}}(x)$ *by*

$$\phi^{\text{even}}(x) = \begin{cases} \phi(x), & x \in [0, L], \\ \phi(-x), & x \in [-L, 0]. \end{cases}$$

(The conditions on $\phi(0)$ and $\phi'(0)$ are needed to make the extensions continuous and differentiable, so that they match the hypothesis of Theorem 1. But you shouldn't worry about them too much because in fact the "full version" of the theorem works under weaker assumptions.)

When the function $\phi^{\text{odd}}(x)$ is well-defined, the theorem tells us that ϕ^{odd} equals its Fourier series. But, since this function is odd by construction, we can apply Lemma 3.1 to deduce that all the a_n coefficients are zero. Therefore,

$$\phi^{\text{odd}}(x) = \text{FS}_{\phi^{\text{odd}}}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right), \qquad x \in (-L, L),$$

where b_n is given by (2.4). In the same way, if we apply first Theorem 1 and then Lemma 3.1 to the even function ϕ^{even}, we find that it has the representation

$$\phi^{\text{even}}(x) = \text{FS}_{\phi^{\text{even}}}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right), \qquad x \in (-L, L),$$

where a_0 and a_n are given by (2.2)–(2.3).

But here comes the trick: Both $\phi^{\text{odd}}(x)$ and $\phi^{\text{even}}(x)$ are equal to $\phi(x)$ for $x \in [0, L]$! This is simply by definition of the odd and even extensions. In conclusion, we have derived a representation of $\phi(x)$, $x \in (0, L)$, in terms of only sines (called the **Fourier sine series**) as well as a representation in terms of only cosines (called the **Fourier cosine series**). We summarize this in the following theorem:

Theorem 2. Let $\phi(x)$, $x \in [0, L]$, be a (differentiable) function. Then

(i) if $\phi'(0) = 0$ then $\phi(x)$ equals its **Fourier cosine series**:

$$\phi(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

for all $x \in (0, L)$, where $a_0 = \frac{1}{L}\int_0^L \phi(x)dx$ and $a_n = \frac{2}{L}\int_0^L \phi(x)\cos\left(\frac{n\pi}{L}x\right)dx$ for $n \geq 1$;

(ii) if $\phi(0) = 0$ then $\phi(x)$ equals its **Fourier sine series**:

$$\phi(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right),$$

for all $x \in (0, L)$, where $b_n = \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{n\pi}{L}x\right)dx$ for $n \geq 1$.

4 General solution of the IBVPs

We can now easily derive a general solution for our three IBVPs, where we now allow any initial condition $\phi(x)$. Let us illustrate this with IBVP (1):

$$\begin{cases} u_t = \alpha^2 u_{xx} & \text{in } \Omega = (0, L) \times (0, \infty), & \text{(PDE)} \\ u(0, t) = u(L, t) = 0 & \text{for all } t > 0, & \text{(BC)} \\ u(x, 0) = \phi(x) & \text{for all } x \in [0, L]. & \text{(IC)} \end{cases}$$

Recall that we know what the solution is if $\phi(x)$ is expressed in terms of sines. Therefore, we use Theorem 2 to represent $\phi(x)$ using its Fourier sine series:

$$\phi(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right).$$

For this initial condition we know that the solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right).$$

We can solve IBVP (2) and IBVP (3) in the same way, but instead of using the Fourier sine series, we use the Fourier cosine series (Theorem 2), respectively the Fourier series (Theorem 1). To summarize, the solutions are as follows:

The general solution of IBVP (1) for heat flow on a laterally insulated rod with fixed boundary conditions,

$$\begin{cases} u_t = \alpha^2 u_{xx} & \text{in } \Omega = (0, L) \times (0, \infty), & \text{(PDE)} \\ u(0, t) = u(L, t) = 0 & \text{for all } t > 0, & \text{(BC)} \\ u(x, 0) = \phi(x) & \text{for all } x \in [0, L], & \text{(IC)} \end{cases}$$

is given by

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right),$$

where

$$b_n = \frac{2}{L} \int_0^L \phi(x) \sin\left(\frac{n\pi}{L} x\right) dx, \qquad n \geq 1.$$

The general solution of IBVP (2) for heat flow on a completely insulated rod,

$$\begin{cases} u_t = \alpha^2 u_{xx} & \text{in } \Omega = (0, L) \times (0, \infty), & \text{(PDE)} \\ u_x(0, t) = u_x(L, t) = 0 & \text{for all } t > 0, & \text{(BC)} \\ u(x, 0) = \phi(x) & \text{for all } x \in [0, L], & \text{(IC)} \end{cases}$$

is given by

$$u(x, t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi}{L} x\right),$$

where

$$a_0 = \frac{1}{L} \int_0^L \phi(x) dx,$$

$$a_n = \frac{2}{L} \int_0^L \phi(x) \cos\left(\frac{n\pi}{L} x\right) dx, \qquad n \geq 1.$$

The general solution of IBVP (3) for heat flow on an insulated circular wire,

$$\begin{cases} u_t = \alpha^2 u_{xx} & \text{in } \Omega = (-L, L) \times (0, \infty), & \text{(PDE)} \\ u(-L,t) = u(L,t) & & \\ u_x(-L,t) = u_x(L,t) & \text{for all } t > 0, & \text{(BC)} \\ u(x,0) = \phi(x) & \text{for all } x \in [-L, L], & \text{(IC)} \end{cases}$$

is given by

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \left(a_n \cos\left(\frac{n\pi}{L} x\right) + b_n \sin\left(\frac{n\pi}{L} x\right) \right)$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} \phi(x) dx,$$
$$a_n = \frac{1}{L} \int_{-L}^{L} \phi(x) \cos\left(\frac{n\pi}{L} x\right) dx, \qquad n \geq 1,$$
$$b_n = \frac{1}{L} \int_{-L}^{L} \phi(x) \sin\left(\frac{n\pi}{L} x\right) dx, \qquad n \geq 1.$$