Analysis III for D-BAUG, Fall 2018 — Lecture 6

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1 D'Alembert's solution of the wave equation

We'll following **IVP (initial value problem)**, which models the vibrations of a very long (in fact, infinite) string. Since the string is infinitely long, it has no endpoints, so our problem has no boundary conditions. That's we call it an IVP, rather than an IBVP.

Find $u = u(x,t)$ such that

$$(1) \qquad \begin{cases} u_{tt} = c^2 u_{xx} & \text{in } \Omega = \mathbb{R} \times (0,\infty), \qquad \text{(PDE)} \\ u(x,0) = f(x) & \\ u_t(x,0) = g(x) & \text{for all } x \in \mathbb{R}. \qquad \text{(IC)} \end{cases}$$

Here $c > 0$ is a given constant, and $f(x)$ and $g(x)$ are given functions.

Just as last time, $f(x)$ describes the initial shape of the string at time $t = 0$, and $g(x)$ describes its initial velocity. D'Alembert obtained the following remarkable result that a solution not only exists, but can be written down directly in terms of the initial conditions—no infinite series or Fourier coefficients necessary!

Theorem (D'Alembert). The general solution of IVP (1) for vibrations of an infinite string is given by

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy$$

This theorem makes clear that the constant c corresponds to speed of wave propagation, at least in cases where $g(x) = 0$. We'll discuss this more later. First, let us verify that the theorem is correct.

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Verification of the theorem. Simply compute the derivatives:

$$u_x = \frac{1}{2}\left(f'(x-ct) + f'(x+ct)\right) + \frac{1}{2c}\left(g(x+ct) - g(x-ct)\right),$$
$$u_{xx} = \frac{1}{2}\left(f''(x-ct) + f''(x+ct)\right) + \frac{1}{2c}\left(g'(x+ct) - g'(x-ct)\right),$$

and

$$u_t = \frac{1}{2}\left(-cf'(x-ct) + cf'(x+ct)\right) + \frac{1}{2c}\left(cg(x+ct) + cg(x-ct)\right),$$
$$u_{tt} = \frac{1}{2}\left(c^2 f''(x-ct) + c^2 f''(x+ct)\right) + \frac{1}{2c}\left(c^2 g'(x+ct) - c^2 g'(x-ct)\right).$$

Clearly $u_{tt} = c^2 u_{xx}$, which shows that $u(x,t)$ satisfies the PDE. It only remains to check the initial conditions:

$$u(x,0) = \frac{f(x-0) + f(x+0)}{2} + 0 = f(x)$$

and

$$u_t(x,0) = \frac{1}{2}\left(-cf'(x-0) + cf'(x+0)\right) + \frac{1}{2c}\left(cg(x+0) + cg(x-0)\right) = g(x),$$

as required. □

So we see that the theorem is correct. However, just verifying its correctness doesn't tell us anything about how to *derive* the solution. Knowing how to derive the solution might help us solve other similar problems, where the theorem itself does not apply. In the next section we will derive the solution of IVP (1) using a method that we have not used before: change of variables.

2 Change of variables and the chain rule

The idea for solving IVP (1) is to change from the original variables (x,t) to new variables (ξ,η) such that in the new variables, the PDE has a simpler form. In fact, we look for variables (ξ,η) such that

$$u_{tt} = c^2 u_{xx} \qquad \Longleftrightarrow \qquad u_{\xi\eta} = 0. \tag{2.1}$$

The following change of variables turns out to accomplish this:

$$\xi = x - ct, \qquad \eta = x + ct.$$

We now need to express the derivative u_{xx} and u_{tt} in terms of derivatives with respect to the new variables (ξ,η). This is done using the chain rule. For instance:

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \times \underbrace{\frac{\partial \xi}{\partial x}}_{=1} + \frac{\partial u}{\partial \eta} \times \underbrace{\frac{\partial \eta}{\partial x}}_{=1} = u_\xi + u_\eta. \tag{2.2}$$

Note that we mixed two different kinds of notation for the partial derivatives here, since the "$\partial/\partial x$" type notation is often useful when dealing with the chain rule. Similarly, the derivative with respect to t can be expressed as:

$$u_t = \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi} \times \underbrace{\frac{\partial \xi}{\partial t}}_{=-c} + \frac{\partial u}{\partial \eta} \times \underbrace{\frac{\partial \eta}{\partial t}}_{=c} = -cu_\xi + cu_\eta. \tag{2.3}$$

Next, we consider the second derivatives:

$$\begin{aligned} u_{xx} = \frac{\partial (u_x)}{\partial x} &= \frac{\partial (u_x)}{\partial \xi} \times \underbrace{\frac{\partial \xi}{\partial x}}_{=1} + \frac{\partial (u_x)}{\partial \eta} \times \underbrace{\frac{\partial \eta}{\partial x}}_{=1} \\ &= \frac{\partial (u_\xi + u_\eta)}{\partial \xi} + \frac{\partial (u_\xi + u_\eta)}{\partial \eta} && \text{(using (2.2))} \\ &= u_{\xi\xi} + u_{\eta\xi} + u_{\xi\eta} + u_{\eta\eta} \\ &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}. && \text{(since } u_{\eta\xi} = u_{\xi\eta}) \end{aligned}$$

Similarly,

$$\begin{aligned} u_{tt} = \frac{\partial (u_t)}{\partial t} &= \frac{\partial (u_t)}{\partial \xi} \times \underbrace{\frac{\partial \xi}{\partial t}}_{=-c} + \frac{\partial (u_t)}{\partial \eta} \times \underbrace{\frac{\partial \eta}{\partial t}}_{=c} \\ &= -c\,\frac{\partial (-cu_\xi + cu_\eta)}{\partial \xi} + c\,\frac{\partial (-cu_\xi + cu_\eta)}{\partial \eta} && \text{(using (2.3))} \\ &= c^2 u_{\xi\xi} - c^2 u_{\eta\xi} - c^2 u_{\xi\eta} + c^2 u_{\eta\eta} \\ &= c^2 (u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}). && \text{(since } u_{\eta\xi} = u_{\xi\eta}) \end{aligned}$$

What did we achieve so far? We have simply established the two formulas

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \qquad \text{and} \qquad u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}).$$

These formulas give us the identity

$$u_{tt} - c^2 u_{xx} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) - c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) = -4c^2 u_{\xi\eta}. \tag{2.4}$$

Therefore, if (PDE) holds, the left-hand side of (2.4) is zero. Then so is the right-hand side. We may divide by $-4c^2$, and find that $u_{\xi\eta} = 0$. In the opposite direction, suppose $u_{\xi\eta} = 0$. Then the right-hand side of (2.4) is zero, and hence also the left-hand side. This tells us that (PDE) holds. Combining these two statements we see that (2.1) is true.

3 Solving the transformed equation

We are now faced with the task of finding the solution of the PDE $u_{\xi\eta}$. For once this is easy!

(1) Integrate with respect to η: Since $u_{\xi\eta} = 0$, we must have that u_ξ is constant in η. That is, $u_\xi = C(\xi)$ for some function $C(\xi)$ which is constant in η but may depend on ξ.

(2) Integrate with respect to ξ: Since $u_\xi = C(\xi)$, we must have that $u = A(\xi) + B(\eta)$, where $A(\xi) = \int C(\xi) d\xi$ is a primitive function of $C(\xi)$, and $B(\eta)$ is some function which is constant in ξ but may depend on η.

Conclusion: The solution u, given in terms of the variables (ξ, η), is of the form

$$u(\xi, \eta) = A(\xi) + B(\eta)$$

for some functions $A(\xi)$ and $B(\eta)$.

Remark 3.1. We could as well have started by integrating with respect to ξ, and then integrate with respect η. You can check that this would not have affected the final form of $u(\xi, \eta)$.

4 Change back to the original variables and match (IC)

To complete the derivation of D'Alembert's solution, we need to express u in the original variables (x, t), and then match the initial conditions. Writing the solution in terms of (x, t) is easy:

$$u(x,t) = u(\xi(x,t), \eta(x,t)) = A(x - ct) + B(x + ct),$$

where A and B are some arbitrary functions. Already here you can see the interpretation of c: The solution is now expressed as a superposition of the two waveforms $A(x)$ and $B(x)$, traveling in opposite directions at a constant speed c.

We now match the initial conditions.

- **First (IC):** $u(x, 0) = f(x)$. This just gives us the equation

$$A(x) + B(x) = f(x). \tag{4.1}$$

- **Second (IC):** $u_t(x, 0) = g(x)$. This gives us the equation

$$-cA'(x) + cB'(x) = g(x),$$

  or, dividing by c, the equation $-A'(x) + B'(x) = \frac{1}{c} g(x)$. Integrating both sides from 0 to x gives us

$$-A(x) + B(x) = \frac{1}{c} \int_0^x g(y) dy + K, \tag{4.2}$$

  where K is a constant. Its value is $K = -A(0) + B(0)$, although this is not going be relevant for us.

- **Combine the equations:** The two equations (4.1)–(4.2) form a linear system, where we treat $A(x)$ and $B(x)$ as unknowns. By first adding the equations and then subtracting them, we obtain

$$2B(x) = f(x) + \frac{1}{c} \int_0^x g(y) dy + K,$$
$$2A(x) = f(x) - \frac{1}{c} \int_0^x g(y) dy - K.$$

This finally gives us the D'Alembert solution:

$$\begin{aligned}
u(x,t) &= A(x-ct) + B(x+ct) \\
&= \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\left(\int_0^{x+ct} g(y)dy - \int_0^{x-ct} g(y)dy + K - K\right) \\
&= \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\left(\int_0^{x+ct} g(y)dy + \int_{x-ct}^{0} g(y)dy\right) \\
&= \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy.
\end{aligned}$$

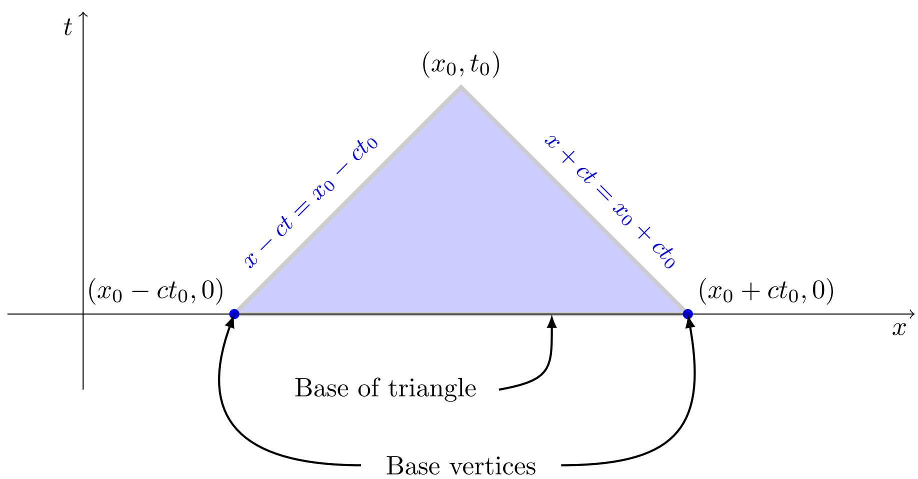
Remark 4.1. The (ξ, η) variables are called **characteristic coordinates**. A similar change-of-variable technique can be used to solve more complicated one-dimensional hyperbolic PDE, even with non-constant coefficients. The goal is always to reduce the original PDE to a transformed equation that is easy to solve.

Remark 4.2. It is **very rare** that a PDE can be solved this explicitly: Here we obtained a completely explicit finite formula for the solution in terms of the initial conditions. This is much nicer than the IBVPs we've looked at so far, where we were only able to express the general solution as an infinite series.

5 Characteristics

Next, we will have a look at the solution in the (x,t) plane to understand it better.

Definition 5.1. *Fix a point (x_0, t_0), $x_0 \in \mathbb{R}$, $t_0 > 0$. The* **characteristic lines** *through (x_0, t_0) are the line going through (x_0, t_0) and $(x_0 - ct_0, 0)$, along with the line going through (x_0, t_0) and $(x_0 + ct_0, 0)$.*



Recall D'Alembert's solution:

$$u(x_0,t_0) = \frac{f(x_0-ct_0)+f(x_0+ct_0)}{2} + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(y)dy.$$

Observe that $2ct_0$ is the length of base of the triangle in the figure above. Therefore, the quantity $\frac{1}{2ct_0}\int_{x_0-ct_0}^{x_0+ct_0} g(y)dy$ gives the average value of g over the base of the triangle. This gives the following alternative way of expressing the solution:

$$u(x_0,t_0) = \left(\begin{array}{l}\text{average value of } f \text{ over}\\ \text{the base vertices}\end{array}\right) + t_0 \cdot \left(\begin{array}{l}\text{average value of } g \text{ over}\\ \text{the base of the triangle}\end{array}\right)$$

In particular, the value $u(x_0,t_0)$ of the solution at x_0, t_0 is completely determined by the values of the initial conditions $f(x)$ and $g(x)$ along the base vertices and the base of the triangle, respectively. It is completely independent of the values of $f(x)$ and $g(x)$ at other points.