D-MAVT
D-MATL

Prof. A. Iozzi
ETH Zürich
Autumn 2018

Analysis III

Serie 1 - preliminary exercises

<u>What's needed:</u> Integration by parts, partial fraction decomposition, polynomial division, linear differential equations, trigonometric formulas.
The techniques needed to solve these exercises are considered as a basic requirement for the whole course.

1. Compute the following primitive integrals using partial integration.

a) $\int \cos(2x)\cos(x)\,dx$

b) $\int \sin^2(x)\,dx$

c) $\int \sin(nx)\sin(mx)\,dx, \quad n \neq m, \quad n, m \neq 0$

Note: it may be necessary to integrate by parts more than once.

2. Find the primitive functions of the followings.

a) $\dfrac{2x^2+3}{x(x-1)^2}$

b) $\dfrac{x^5+2}{x^2-1}$

c) $\dfrac{1}{1+x^3}$

Hint: it may be useful to find an alternate expression of these functions first.

3. Solve the following linear differential equations:

a) $y''(t) + y'(t) - 6y(t) = 0$

b) $4y''(t) + 12y'(t) + 9y(t) = 0$

c) $y''(t) - 6y'(t) + 13y(t) = 0$

4. **a)** With the help of trigonometric addition formulas, verify that the following identity is true

$$\cos(\alpha x)\cos(\beta x) = \frac{1}{2}\left(\cos((\alpha+\beta)x) + \cos((\alpha-\beta)x)\right), \qquad \text{for } \alpha, \beta, x \in \mathbb{R}$$

b) Find similar expressions for $\sin(\alpha x)\sin(\beta x)$ and $\sin(\alpha x)\cos(\beta x)$.

c) Prove for n, $m \in \mathbb{N}$ the orthogonality relation

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} 0, & \text{if } n \neq m \\ \pi, & \text{if } n = m \quad \text{and } n, m \neq 0 \\ 2\pi, & \text{if } n = m = 0. \end{cases}$$

5. (Bonus exercise) Solutions of this exercise may not be provided in full but just sketched or some strong hints will be given.

a) Find a primitive of the secant function:

$$\int \sec(x)\,dx,$$

remember that $\sec(x) = 1/\cos(x)$.

b) Prove that[1]

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2} = \frac{22}{7} - \pi.$$

[1] *Historical fact:* Because the function we are integrating is a positive function, we get that $22/7 > \pi$. This was not obvious at all in the antiquity (indeed, looking at the expansion in decimals, these two numbers are very close: $22/7 = 3,\overline{142857}, \pi = 3,141592\ldots$). In fact its first mathematical proof was shown by Archimedes, who certainly didn't know what a primitive of a function was, but certainly was a skilled mathematician.

c) The Gamma function is the function

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x}\, dx, \qquad \alpha > 0.$$

First, one should note that the function is well defined, that is the integral actually converges for each value of $\alpha > 0$.

(i) Prove that for each $\alpha > 0$ the following recursion formula holds:

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

(ii) Compute explicitely $\Gamma(1) = 1$.

(iii) Deduce from (i) and (ii) that for integer numbers $n \in \mathbb{N}$:

$$\Gamma(n+1) = n!$$

Therefore the Gamma function is a possible extension of the factorial function to all positive real numbers.