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Analysis III

Autumn 2018

Serie 10

What's needed: Normal form of a PDE, D'Alembert solution of the wave equation.

1. Consider the following PDE:

$$u_{xx} - 4u_{xy} + 3u_{yy} = 0.$$

Determine its type (hyperbolic, parabolic or elliptic), bring it in normal form with an opportune change of coordinates, and give all possible solutions.

2. Let $u(x,t)$ be the solution of the following problem (1-dimensional wave equation on the line).

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x), & x \in \mathbb{R} \\ u_t(x,0) = 0, & x \in \mathbb{R} \end{cases}$$

where

$$f(x) = \begin{cases} e^{\frac{x^2}{x^2-1}}, & |x| < 1 \\ 0, & \text{otherwise.} \end{cases}$$

a) Sketch a graph of $f(x)$, which is the solution at the initial time.

b) Sketch a graph of the solution at the time $t = 2, u(x,2)$.

c) Prove that, for each fixed $x \in \mathbb{R}$:

$$\lim_{t \to +\infty} u(x, t) = 0.$$

Give an explanation why this is always true if we start from

$$\begin{cases} f(x) \text{ such that: } \displaystyle\lim_{|x| \to +\infty} f(x) = 0, \\ g(x) = 0. \end{cases}$$

Hint: you need to use d'Alembert formula. You are not required to write down explicitely the solution $u(x, t)$ for each time, but it's important that you have in mind how the solution evolves with the changing of t.

3. Let $u(x, t)$ be the solution of the problem

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\ t > 0 \\ u(x, 0) = f(x) = \begin{cases} 1, & |x| \leqslant 1 \\ 0, & |x| > 1 \end{cases}, & x \in \mathbb{R} \\ u_t(x, 0) = g(x) = \begin{cases} 1, & |x| \leqslant 1 \\ 0, & |x| > 1 \end{cases}, & x \in \mathbb{R} \end{cases}$$

a) Find the values $u(0, \frac{1}{2})$ and $u(\frac{3}{2}, \frac{1}{2})$.

b) Find, for each fixed $x \in \mathbb{R}$, the limit

$$\lim_{t \to +\infty} u(x, t).$$

Hint: use d'Alembert formula, and you don't need to find an explicit formula for $u(x, t)$ for each x and t.

Due by: Thursday 29 / Friday 30 November 2018.