D-MAVT
D-MATL

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ETH Zürich
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Analysis III

Serie 11

What's needed: Fourier series solution of the 1-dimensional heat equation and Laplace equation on a rectangle (steady 2-dimensional heat equation).

1. Find, via Fourier series, the solution of the 1-dimensional heat equation with the following initial condition:

$$\begin{cases} u_t = 4\,u_{xx}, & \\ u(0,t) = u(1,t) = 0, & t \geqslant 0 \\ u(x,0) = f(x), & 0 \leqslant x \leqslant 1 \end{cases}$$

where

$$f(x) = \sin(\pi x) + \sin(5\pi x) + \sin(10\pi x).$$

2. An aluminium bar of length $L = 1(m)$ has thermal diffusivity of (around)[1]

$$c^2 = 0.0001\left(\frac{m^2}{sec}\right) = 10^{-4}\left(\frac{m^2}{sec}\right).$$

It has initial temperature given by $u(x,0) = f(x) = 100\sin(\pi x)$ (°C), and its ends are kept at a constant 0°C temperature. Find the first time t^* for which the whole bar will have temperature $\leqslant 30$°C.
In mathematical terms, solve

$$\begin{cases} u_t = 10^{-4} u_{xx}, & \\ u(0,t) = u(1,t) = 0, & t \geqslant 0 \\ u(x,0) = 100\sin(\pi x), & 0 \leqslant x \leqslant 1. \end{cases}$$

and find the smallest t^* for which

$$\max_{x\in[0,1]} u(x,t^*) \leqslant 30.$$

[1] we are approximating the standard value which would be $c^2 \approx 0.000097 m^2/sec$ to make computations easier.

3. Solve the following Laplace equation (steady heat equation) on the rectangle

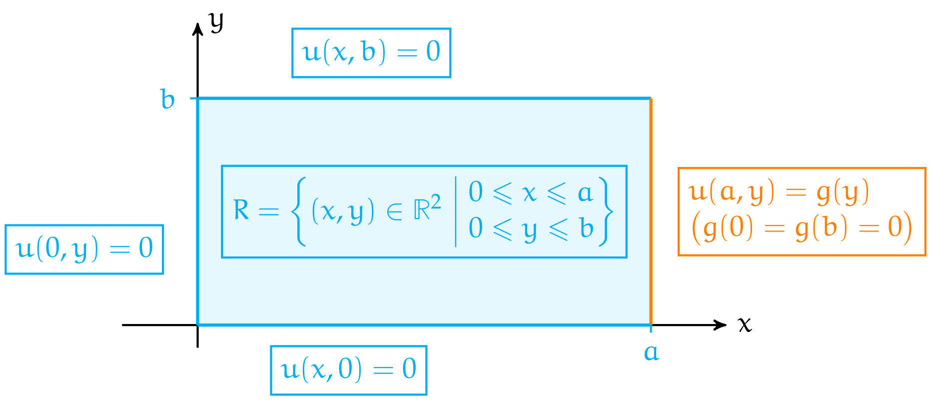
$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 2\},$$

$$\begin{cases} \Delta u = 0, & (x, y) \in R \\ u(0, y) = u(1, y) = 0, & 0 \leqslant y \leqslant 2 \\ u(x, 0) = 0, & 0 \leqslant x \leqslant 1 \\ u(x, 2) = f(x), & 0 \leqslant x \leqslant 1 \end{cases}$$

where

$$f(x) = x(1 - x).$$

4. Adapt the method used to solve the previous Laplace equation in the case in which the only nontrivial initial boundary condition is on the right vertical segment of the rectangle



$$\begin{cases} \Delta u = 0, & (x, y) \in R \\ u(x, 0) = u(x, b) = 0, & 0 \leqslant x \leqslant a \\ u(0, y) = 0, & 0 \leqslant y \leqslant b \\ u(a, y) = g(y), & 0 \leqslant y \leqslant b \end{cases}$$

where $g(y)$ is any function with prescribed boundary conditions

$$g(0) = g(b) = 0.$$

Due by: Thursday 6 / Friday 7 December 2018.