Analysis III

Serie 12

What's needed: Review/variation of the Fourier series solution of the 1-dimensional heat equation. Integral solution of the 1-dimensional heat equation, wave equation on a rectangular membrane (optional).

1. (Old exercise)

a) Using the method of separation of variables, find a Fourier series solution for the following problem:

$$\begin{cases} u_t = c^2 u_{xx}, & 0 \leqslant x \leqslant L, t \geqslant 0 \\ u(0,t) = u(L,t) = 0, & t \geqslant 0 \\ u_x(x,0) = h(x), & 0 \leqslant x \leqslant L \end{cases}$$

where $h(x)$ is any (differentiable) function such that

$$\int_0^L h(x)\,dx = 0.$$

Why do we need to require this condition?

b) Find the general solution for the following problem:

$$\begin{cases} u_t = c^2 u_{xx}, & 0 \leqslant x \leqslant L, t \geqslant 0 \\ u(0,t) = a, & t \geqslant 0 \\ u(L,t) = b, & t \geqslant 0 \\ u(x,0) = f(x), & 0 \leqslant x \leqslant L \end{cases}$$

where $a, b \in \mathbb{R}$ are arbitrary constants, and $f(x)$ is any (twice differentiable) function such that $f(0) = a, f(L) = b$.
Compute, for each fixed $0 \leqslant x \leqslant L$, the asymptotic limit

$$\lim_{t \to +\infty} u(x,t).$$

Hint: you may want to perform a substitution of the form $u = v + l$ and solve an easier, corresponding problem for v.

2. Consider the following 1-dimensional heat equation, with initial temperature given by a gaussian distribution

$$\begin{cases} u_t = c^2 u_{xx}, & x \in \mathbb{R}, t \geqslant 0 \\ u(x,0) = e^{-x^2}, & x \in \mathbb{R} \end{cases}$$

a) Find an explicit formula for the solution $u(x,t)$ (explicit = no unsolved integrals or other implicit computations left).
Hint: the integral formula gives a very complicated integral to compute. It's better to solve directly via Fourier transform as the procedure is very much simplified for this particular choice of $f(x)$ (use the results of Exercises **5.a)** and **6.** in Serie 7).

b) Say if the following equalities are true or false.

(i) $u\left(0, \frac{1}{4c^2}\right) = \frac{\sqrt{2}}{2}$ (T) (F)

(ii) $u(1,1) = \frac{1}{\sqrt{1+4c^2}} e^{\frac{1}{1+4c^2}}$ (T) (F)

(iii) $u(0,1) = \frac{1}{\sqrt{1+4c^2}}$ (T) (F)

c) Write explicitely the function $\varphi(t) := u(0,t)$ which describes, for $t \geqslant 0$, the evolution of the temperature in the point $x = 0$, and sketch a graph of it.

d) Find, for each each temperature $0 < \lambda < 1$, the only time $t = t(\lambda)$ for which the point at the origin $x = 0$ will have temperature equal to λ.

3. **(Bonus exercise - not treated in the lecture)**
An elastic membrane of squared shape of side length 1 m is let vibrating from the initial position described by the function $f(x, y)$ below, with initial speed zero. The material of which the membrane is composed is such that its vibrating waves will propagate with speed $c = 1\,\mathrm{m/s}$.
In mathematical terms the profile of the membrane at the time t is described by the function $u(x, y, t)$ which is the solution of the following problem

$$R := \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leqslant x, y \leqslant 1\}$$

$$\begin{cases} u_{tt} = u_{xx} + u_{yy}, & (x, y) \in R, \quad t \geqslant 0 \\ u(x, 0, t) = u(x, 1, t) = u(0, y, t) = u(1, y, t) = 0, & 0 \leqslant x, y \leqslant 1, \; t \geqslant 0 \\ u(x, y, 0) = f(x, y) = \sin(\pi x)\sin(2\pi y), & (x, y) \in R \\ u_t(x, y, 0) = 0. & (x, y) \in R \end{cases}$$

a) Find the solution $u(x, y, t)$.
Hint: Use the formula at p. 65 of the Lecture notes, and impose the initial conditions yourself.

b) What is the height range in which the membrane vibrates? That is, find

$$\text{height}_{\text{min}} := \min_{\substack{(x,y)\in R \\ t\geqslant 0}} u(x, y, t), \qquad \text{height}_{\text{max}} := \max_{\substack{(x,y)\in R \\ t\geqslant 0}} u(x, y, t)$$

c) Find the instants in which the membrane is completely flat. That is, find the $t^* \geqslant 0$ such that

$$u(x, y, t^*) = 0, \quad \forall (x, y) \in R.$$

d) Find the instants in which the membrane is momentaneously still. That is, find the $t^* \geqslant 0$ such that

$$u_t(x, y, t^*) = 0, \quad \forall (x, y) \in R.$$

e) Is the vibration of the membrane periodic? If yes, what is the fundamental period?

Due by: Thursday 13 / Friday 14 December 2018.