D-MAVT
D-MATL

Prof. A. Iozzi
ETH Zürich
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Analysis III

Serie 13

What's needed: Derivatives in polar coordinates, Laplacian in polar coordinates, Laplace equation on a disk with prescribed boundary conditions (Dirichlet problem).

1. Find the solution of the 2-dimensional Laplace equation (steady heat equation)

$$\nabla^2 u = 0$$

on the disk $D = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leqslant 4\}$ with prescribed temperature on the external circumference given by

$$u(x, y) = x^3, \qquad (x, y) \in \partial D \left(= \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 4\} \right).$$

[You can use the trigonometric formula

$$\cos^3(\theta) = \frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta) \,]$$

Do as follows:

a) Write the boundary condition in polar coordinates.

b) Solve the problem in polar coordinates, using the methods/formulas learnt in the Lecture notes.

c) Express the solution in the standard cartesian coordinates:

$$u(x, y) = ?$$

2. Let $R > 0$.

a) Find the solution $u(r, \theta)$ of the Dirichlet problem on the disk of radius R in polar coordinates:

$$\begin{cases} \nabla^2 u = 0, & 0 \leqslant r \leqslant R, 0 \leqslant \theta \leqslant 2\pi \\ u(R, \theta) = \sin^2(\theta), & 0 \leqslant \theta \leqslant 2\pi. \end{cases}$$

[You can use the trigonometric formula

$$\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta) \,]$$

b) Find the maximum of $u(r, \theta)$. In which point(s) is it reached?

c) Express the solution in the standard cartesian coordinates.

3. Let u be the unique harmonic function ($\nabla^2 u = 0$) on the unit disk which on the boundary is

$$u(x, y) = xy + 3.$$

Without computing any integral or using any formula from the script, answer the following questions:

a) Find the solution in the standard coordinates $u(x, y)$.

b) What's the value in the center of the disk $u(0, 0)$= ?

c) What's the maximum of u and in which point(s) is it reached?

4. Prove, without computing explicitely the integrals, that for each $0 \leqslant r < 1$ and for each $0 \leqslant \theta \leqslant 2\pi$:

a)

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2} \, d\phi = 1.$$

b)

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2)\big(\cos^3(\phi)\sin(\phi) - \sin^3(\phi)\cos(\phi)\big)}{1 - 2r\cos(\theta - \phi) + r^2} \, d\phi = \frac{r^3}{4}\sin(4\theta).$$