D-MAVT
D-MATL

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Analysis III

Serie 14

What's needed: Properties of harmonic functions (mean value property and maximum principle). Well-posed and ill-posed problems.

Notation: for any positive radius $R > 0$, D_R is the closed disk of radius R and ∂D_R its boundary.

1. Let u the unique harmonic function on the disk of radius R which on the boundary is

$$u(x, y) = x^2 y^2, \qquad (x, y) \in \partial D_R.$$

Answer, *without finding explicitely the function on the whole disk*, the following questions.

a) What's the value in the center of the disk $u(0, 0) = ?$

b) What's the maximum of u and in which point(s) is it reached?

c) Same question for the minimum.

2. For each of the following problems, determine whether they admit a (at least one) solution or not.

a)

$$\begin{cases} \nabla^2 u = 0, & \text{in } D_R \\ u = x^a y^b, & \text{on } \partial D_R \\ u(0, 0) = 0. \end{cases}$$

where $a, b \geqslant 0$ are integer numbers.
Hint: the problem is to understand which choice of parameters a, b is compatible with the other data. You should find out it has to do with their parity.

b)

$$\begin{cases} \nabla^2 u = 0, & \text{in } D_R \\ u(R,\theta) = 3R\, e^{\frac{(\theta-\pi)^2}{\theta(\theta-2\pi)}}, & 0 \leqslant \theta \leqslant 2\pi \ \big(\text{ parametrising } \partial D_R\big) \\ \max\limits_{D_R} u = \pi \end{cases}$$

c)

$$\begin{cases} \nabla^2 u = 0, & \text{in } D_R \\ u(R,\theta) = \sin^9(\theta), & 0 \leqslant \theta \leqslant 2\pi \ \big(\text{ parametrising } \partial D_R\big) \\ u + 1 \geqslant 0, & \text{in } D_R \end{cases}$$

3. Consider the following Neumann problem (Laplace equation with fixed normal derivative on the boundary):

$$\begin{cases} \nabla^2 u = 0, & \text{in } D_R \\ \partial_n u(R,\theta) = \theta(2\pi-\theta)(\theta^2-12), & 0 \leqslant \theta \leqslant 2\pi \ \big(\text{ parametrising } \partial D_R\big) \end{cases}$$

a) Is there a solution?

b) If the answer is yes, how many?