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Analysis III

Serie 15 - Ferienserie

What's needed: Everything.
Rules: Every step and computation should be justified. You are allowed to use trigonometric identities and Laplace transforms tables. You are also allowed to use formulas from the script when you're told so.

1. Find the solution $y = y(t)$ of the following IVP:

$$\begin{cases} y''' + y'' = \delta(t-6), & t \geqslant 0 \\ y(0) = y'(0) = 0, \\ y''(0) = 1. \end{cases}$$

[You may use the identity

$$\frac{1}{s^2(s+1)} = \frac{1}{s+1} - \frac{1}{s} + \frac{1}{s^2} \quad]$$

2. Let

$$f(x) = \begin{cases} x, & 0 \leqslant |x| \leqslant 1 \\ -\operatorname{sign}(x), & 1 < |x| < 2 \\ 0, & \text{otherwise.} \end{cases}$$

a) Sketch the graph of this function, at least in the interval $x \in [-3, 3]$.

b) Let

$$F(x) = \frac{2}{\pi} \int_0^{+\infty} \frac{\big(\sin(\omega) - 2\omega\cos(\omega) + \omega\cos(2\omega)\big)\sin(\omega x)}{\omega^2}\, d\omega.$$

Prove that

$$\forall x \in \mathbb{R},\ x \neq \pm 1, \pm 2, \qquad F(x) = f(x).$$

Please turn!

c) What are the values of the integral in the remaining 4 points?

(i) $F(-1) = ?$
(ii) $F(1) = ?$
(iii) $F(-2) = ?$
(iv) $F(2) = ?$

3. Consider a string of length $L = \pi$, whose waves propagate with speed $c = 1$. Let $u(x,t)$ be the height of the string at time $t \geqslant 0$ and point $x \in [0,\pi]$.
The string is initially motionless (zero initial speed), has initial shape given by

$$u(x,0) = x(x-\pi)$$

and fixed endpoints at height zero.

a) Formulate the problem whose solution is $u(x,t)$.

b) Find the solution. You are allowed to use the formula from the script.

4. Let $c > 0$. Consider the following problem:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t \geqslant 0, x \in \mathbb{R} \\ u(x,0) = e^{-x^2}\sin^2(x) + x, & x \in \mathbb{R} \\ u_t(x,0) = xe^{-x^2}. & x \in \mathbb{R} \end{cases}$$

a) Find the solution $u(x,t)$. You may use D'Alembert formula.
[Simplify the expression as much as possible: no unsolved integrals].

b) For a fixed $a \in \mathbb{R}$, determine the asymptotic limit

$$\lim_{t\to+\infty} u(a,t).$$

5. **a)** We recall that for every $a > 0$ the Fourier transform of the Gaussian e^{-ax^2} is given by

$$\mathcal{F}\left[e^{-ax^2}\right](\omega) = \frac{1}{\sqrt{2a}} e^{-\frac{\omega^2}{4a}}. \tag{1}$$

Let $a > 0$ be fixed. From (1) deduce the integrals

(i) $\int_{\mathbb{R}} e^{-ax^2}\, dx$

(ii) $\int_{\mathbb{R}} x\, e^{-ax^2}\, dx$

(iii) $\int_{\mathbb{R}} x^2\, e^{-ax^2}\, dx$.

[You can use the formula

$$\mathcal{F}\left[x^k f(x)\right](\omega) = i^k \frac{d^k}{d\omega^k} \mathcal{F}\left[f(x)\right](\omega)\Bigg] \tag{2}$$

b) Recall that the solution $u = u(x,t)$ of the following problem

$$\begin{cases} u_t = u_{xx}, & t \geqslant 0, x \in \mathbb{R} \\ u(x,0) = f(x), & x \in \mathbb{R} \end{cases} \tag{3}$$

is given by

$$u(x,t) = \int_{-\infty}^{+\infty} f(y) K(x-y,t)\, dy, \tag{4}$$

where

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \tag{5}$$

is the so-called *Heat Kernel*.

Compute explicitely the solution of the problem

$$\begin{cases} u_t = u_{xx}, & t \geqslant 0, x \in \mathbb{R} \\ u(x,0) = x^2. & x \in \mathbb{R} \end{cases} \tag{6}$$

Hint: use the formula (4) and the results from Exercise **5.a)**.

6. Consider the following problem:

$$\begin{cases} u_t = c^2 u_{xx}, & t \geqslant 0, x \in [0,\pi] \\ u(0,t) = 2, & t \geqslant 0 \\ u(\pi,t) = 3, & t \geqslant 0 \\ u(x,0) = f(x), & x \in [0,\pi] \end{cases} \tag{7}$$

where

$$f(x) = \sin(x) - 3\sin(3x) + \frac{x}{\pi} + 2.$$

The boundary conditions are not homogeneous, therefore one cannot directly apply the formulas known. You should argue as follows:

a) Construct a function $w(x)$ with $w(0) = 2$, $w(\pi) = 3$ and $w'' = 0$.

b) Let u be a solution of the above problem (7). State the corresponding problem solved by the function $v(x,t) := u(x,t) - w(x)$.

c) Solve the problem for v using the method of separation of variables from scratch. Show all the steps of the method of separation of variables.

d) Find the solution u of the original problem (7) .

7. Let $D_1 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leqslant 1\}$ be the centered disk of radius 1 and $u = u(x,y)$ the solution of the Dirichlet problem

$$\begin{cases} \nabla^2 u = 0, & \text{in } D_1 \\ u(x,y) = x^5. & \text{on } \partial D_1 \end{cases}$$

a) Find the value in the center of the disk, $u(0,0)$.

b) Find the maximum of u on the whole disk,

$$\max_{(x,y)\in D_1} u(x,y).$$

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