D-MAVT
D-MATL

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ETH Zürich
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Analysis III

Serie 2

<u>What's needed:</u> Laplace transform (exponential, polynomial, sine and cosine), s-shifting property, inverse Laplace transform.

1. Find the Laplace transform $F(s) := \mathscr{L}(f)(s)$ of the following functions

a) $f(t) = t^2 + 4t + 1$

b) $f(t) = \frac{1}{\sqrt{t}}$, using that

$$\Gamma\left(\frac{1}{2}\right)\left(= \int_0^{+\infty} t^{-1/2} e^{-t}\, dt\right) = \sqrt{\pi}$$

c) $f(t) = \sin(\omega t), \quad \omega \in \mathbb{R}$

d) $f(t) = \cos(\omega t), \quad \omega \in \mathbb{R}$

e) $f(t) = \sin(\alpha t)\cos(\beta t), \quad \alpha, \beta \in \mathbb{R}$

Hint: Exercise **e)** is much simplified using the alternative expression for that function found in Exercise **4.b)** of Serie 1.

2. Given a function $f(t)$ denote its Laplace transform by $F(s) := \mathscr{L}(f(t))(s)$. From the lecture we know that (for sufficiently nice functions) multiplication by t on the time-domain corresponds to derivative on the frequency-domain, that is

$$\mathscr{L}(tf(t))(s) = -\frac{d}{ds}F(s) \tag{1}$$

Using this property, prove that actually for each $n \in \mathbb{N}$:

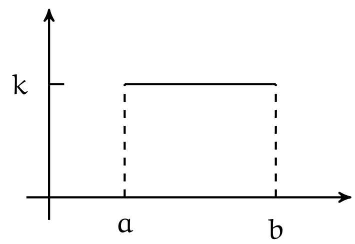
$$\mathscr{L}(t^n f(t))(s) = (-1)^n \frac{d^n}{ds^n}F(s). \tag{2}$$

As an example, for $f(t) = 1$, we will find the known result

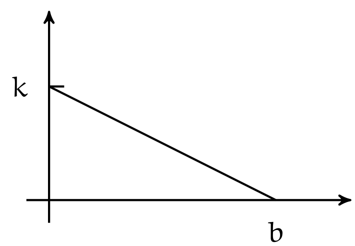
$$\mathscr{L}(t^n)(s) = \frac{n!}{s^{n+1}}.$$

3. Find the Laplace transform of the following functions:

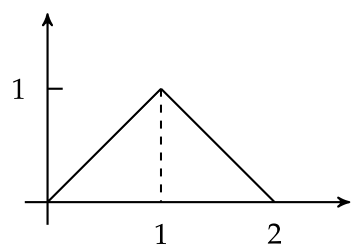
a)



b)



c)



4. Find the <u>inverse</u> Laplace transform $f = \mathscr{L}^{-1}(F)$ of

a) $F(s) = \dfrac{1}{s^4}$

b) $F(s) = \dfrac{1}{(s-8)^{10}}$

c) $F(s) = \dfrac{s+3}{s^2-9}$

d) $F(s) = \dfrac{24}{(s-5)(s+3)}$

e) $F(s) = \dfrac{1}{s^2+4}$

f) **(*)** $F(s) = \dfrac{1}{s^2+4s+20}$

g) **(**)** $F(s) = \dfrac{s+1}{(s+2)(s^2+s+1)}$

h) **(**)** $F(s) = \dfrac{s}{(s-1)^2(s^2+2s+5)}$

where **(*),(**)** denotes the degree of difficulty.
Hint: for exercise **f)** it may be useful to recognize that $s^2+4s+20$ can be written in the form $(s+a)^2+\omega^2$ for an opportune choice of a, ω. Then use the $s-$shifting property. The same technique is needed for **g),h)** but first it's opportune to use partial fraction decomposition to simplify the expression.

5. **(Bonus exercise)**

a) <u>(For those who have never seen this)</u>
<u>Exercise **1.b)** has been asked to solve</u> using that $\Gamma(1/2) = \sqrt{\pi}$, and this exercise proves it. Let's call $I := \Gamma(1/2)$ this value.

(i) Use an opportune change of variables to prove that

$$I = 2\int_0^{+\infty} e^{-x^2}\,dx$$

(ii) Note that

$$2\int_0^{+\infty} e^{-x^2}\,dx = \int_{-\infty}^{+\infty} e^{-x^2}\,dx$$

(iii) Compute this integral in a smart way by computing its square. Fill the dots to get

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2} dy\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy =$$
$$= \cdots = \pi$$

From (iii) must be either $I = \pm\sqrt{\pi}$. But I is obtained by integrating a positive function, therefore it must be the positive value $I = \sqrt{\pi}$.

b) Laplace transform of a finite linear combination of functions is the linear combination of their Laplace transforms:

$$\mathscr{L}\left(a_1 f_1 + \dots a_m f_m\right) = a_1 \mathscr{L}(f_1) + \dots a_m \mathscr{L}(f_m), \quad a_1, \dots, a_m \in \mathbb{R}$$

The same thing is true for infinite linear combination of functions under opportune conditions of convergence which are all satisfied in the following cases. To explain better, let's pretend we don't know the Laplace transform of the exponential and let's compute it explicitely from its power series expression

$$\mathscr{L}(e^{at})(s) = \mathscr{L}\left(\sum_{k=0}^{+\infty} \frac{(at)^k}{k!}\right)(s) = \sum_{k=0}^{+\infty} \frac{a^k}{k!}\mathscr{L}(t^k)(s) =$$
$$= \sum_{k=0}^{+\infty} \frac{a^k}{\cancel{k!}} \cdot \frac{\cancel{k!}}{s^{k+1}} = \frac{1}{s}\sum_{k=0}^{+\infty}\left(\frac{a}{s}\right)^k = \frac{1}{s}\cdot\frac{1}{1-\frac{a}{s}} = \frac{1}{s-a}$$

(i) Find again $\mathscr{L}(\sin(\omega t))(s)$ using the power series expansion

$$\sin(\omega t) = \sum_{k=0}^{+\infty}(-1)^k \frac{(\omega t)^{2k+1}}{(2k+1)!}$$

and verify that the result is the same already found in Exercise **1.c)**.

(ii) Using the same technique, prove that[1]

$$\mathscr{L}\left(\frac{\sin(t)}{t}\right) = \arctan\left(\frac{1}{s}\right)$$

__Due by:__ Thursday 4 / Friday 5 October 2018.

[1] The power series expansion of the arctangent is

$$\arctan(x) = \sum_{k=0}^{+\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)}.$$