D-MAVT
D-MATL

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ETH Zürich
Autumn 2018

Analysis III

Serie 4

What's needed: Dirac delta function and Laplace transform, convolution product with applications to ODEs and integral equations.

Useful remind: The convolution of two functions $f, g : [0, +\infty) \to \mathbb{R}$ is defined as the new function $f \star g : [0, +\infty) \to \mathbb{R}$

$$(f \star g)(t) := \int_0^t f(\tau) g(t-\tau) d\tau$$

One of the most useful property is that

$$\mathcal{L}(f \star g) = \mathcal{L}(f) \cdot \mathcal{L}(g) \tag{1}$$

and, as a consequence of the essential uniqueness of Laplace transform,

$$F = \mathcal{L}(f),\ G = \mathcal{L}(g) \implies \mathcal{L}^{-1}(FG) = f \star g. \tag{2}$$

1. Find the inverse Laplace transform of the following functions.

a) $F(s) = \dfrac{e^{-2s}}{s^2+4}$

b) $F(s) = \dfrac{e^{-s}}{(s+1)^3}$

c) $F(s) = \dfrac{1}{s(s^2+1)}$

d) $F(s) = \dfrac{1}{(s^2+1)^2}$

2. Compute the following convolutions.

a) $e^{at} \star e^{bt} \quad (a, b \in \mathbb{R})$

b) $\sin(t) \star \cos(t)$

Note: Exercise **a)** requires a different discussion for the cases $a \neq b$ and $a = b$.

3. Solve the following IVP:

$$\begin{cases} y'' + 5y' + 6y = \delta(t - \frac{\pi}{2}) + u(t - \pi)\cos(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

4. (Bonus exercise)

a) Use the Laplace transform to verify that the solution of the following IVP

$$\begin{cases} y' = g(t) \\ y(0) = c \end{cases}$$

is the obvious

$$y(t) = c + \int_0^t g(\tau)\,d\tau$$

b) Use the Laplace transform to verify that the solution of the following IVP

$$\begin{cases} y'' + y = g(t) \\ y(0) = c \\ y'(0) = d \end{cases}$$

is

$$y(t) = c\cos(t) + d\sin(t) + \int_0^t \sin(\tau)g(t - \tau)\,d\tau$$

Due by: Thursday 18 / Friday 19 October 2018.