Prof. A. Iozzi
ETH Zürich
Autumn 2018

Analysis III

Serie 5

What's needed: Fundamentals of Fourier analysis, periodic functions, trigonometric polynomials and trigonometric series, Fourier coefficients' formula.

1. A function is periodic of period $P > 0$ if $f(x+P) = f(x)$ for all $x \in \mathbb{R}$. A fundamental period is (if it exists) the smallest positive number P for which f is periodic of period P.
Determine if the following functions are periodic or not.
In the affirmative cases, determine their fundamental period.[1]

a) $f(x) = x$

b) $f(x) = \cos(4x) + \sin(3x)$

c) $f(x) = \sinh(x)$

d) $f(x) = \cos^4(2x)$

e) **(*)** $f(x) = \sin(x^2)$

2. Prove that the Fourier series of $\sin^2(x), \cos^2(x)$, viewed as 2π-periodic functions are, respectively,

$$\sin^2(x) \quad \rightsquigarrow \quad \frac{1}{2} - \frac{1}{2}\cos(2x), \qquad \cos^2(x) \quad \rightsquigarrow \quad \frac{1}{2} + \frac{1}{2}\cos(2x)$$

[1] For example for a positive integer $m \geqslant 1$ the functions $\sin(mx), \cos(mx)$ are periodic with fundamental period $\frac{2\pi}{m}$.

3. Find the Fourier series of the 2L-periodic extension of

$$f(x) = x, \quad x \in [-L, L)$$

4. (Bonus exercise) This exercise relates the periodicity of a function with its derivative(s) and its properties of boundedness.
Let $f : \mathbb{R} \to \mathbb{R}$ be any function.

a) Prove that if f is periodic and continuous, then it is bounded.

b) Prove that if f is differentiable, and periodic of period P, then also f' is periodic with the same period.

c) From **a)** and **b)** deduce that if f is periodic and smooth, then it is bounded and all its derivative are bounded as well.

d) Use **c)** to give a very simple proof that $\sin(x^2)$ is not periodic.[2]

Due by: Thursday 25 / Friday 26 October 2018.

[2] This is much simpler than proving that the function can't be periodic by computing explicitely its values at each possible x and $x + P$. Not to mention that this gets almost impossible when the form is more complicated than just $\sin(x^2)$ while this criteria is always easy to test. For example this proves immediately that all functions like $\sin(p(x)), \cos(p(x))$ with $p(x)$ a polynomial of degree at least 2 are not periodic without doing any computation.