D-MAVT
D-MATL

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Analysis III

Serie 6

What's needed: Pointwise convergence of Fourier series, even and odd functions, half-range expansion, complex Fourier series, approximation by trigonometric polynomials, square error.

1. Consider the function

$$f(x) = \begin{cases} x, & 0 \leqslant x \leqslant \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} \leqslant x \leqslant \pi \end{cases}$$

a) Extend f to an even function on the interval $[-\pi, \pi]$ and then finally to an even, 2π-periodic function on $\mathbb{R}$ and call this function f_e.
Sketch the graph of f_e and find its Fourier series.

b) Do the same for the odd, 2π-periodic extension[1] of f (call this f_o).

2. Consider the function x in the interval $[1, 2]$ and extend it to an even, 2-periodic function f on $\mathbb{R}$.

a) Sketch the graph of f and find its Fourier series.

b) Will the Fourier series converge pointwise to the function f everywhere? (Justify your answer using what learnt in the script).

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[1] to be precise, we can't extend f to an odd, periodic function everywhere. In fact by periodicity and oddness we should have $f_o(-\pi) = f_o(\pi) = -f_o(-\pi)$, and therefore $f_o(\pm\pi) = 0$, while $f(\pi) = \pi$. The points in which there is a doubt about what value to assign to this new function are the odd integer multiples of π. Let's assign to these points the value π just to fix the convention, at the end - as you can observe - nothing will be depend on the choice of this value, and we could have also let f_o not defined.

3. Let f be the 2L-periodic extension of x from $[-L, L)$ to the whole real line as in Exercise **3.** of Serie 5. Find its complex Fourier series

$$\sum_{n=-\infty}^{+\infty} c_n e^{i\frac{n\pi}{L}x}$$

Verify that the coefficients c_n of this series are related to the real coefficients a_n, b_n as in the script.
If you have not computed it before: the real Fourier series of f is

$$\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{2L}{\pi n} \sin\left(\frac{n\pi}{L}x\right) \quad \rightsquigarrow \quad \begin{cases} a_n = 0 \\ b_n = (-1)^{n+1} \frac{2L}{\pi n} \end{cases}$$

4. Consider again the 2L-periodic extension of x as in the previous Exercise. Find the minimum value E_N^* of the square error at the step N, which is

$$E_N^* = \int_{-L}^{L} f^2 \, dx - L\left(2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2)\right), \qquad a_0, a_n, b_n \text{ Fourier coefficients.}$$

(To check that your computation is correct) prove that

$$\lim_{N \to +\infty} E_N^* = \frac{2}{3}L^3 - 4\frac{L^3}{\pi^2} \sum_{n=1}^{+\infty} \frac{1}{n^2}$$

5. Let f be any 2L-periodic function. From the script we know that if f is well behaved (for example everywhere continuous except a discrete set of points and with left and right derivatives at every point) then, calling by F its Fourier series, we have for every point x_0

$$F(x_0) = \frac{1}{2}\left(f^+(x_0) + f^-(x_0)\right), \qquad \text{where } f^\pm(x_0) = \lim_{x \to x_0^\pm} f(x) = \lim_{\epsilon \to 0^+} f(x_0 \pm \epsilon)$$

In particular if f is continuous in x_0 then $F(x_0) = f(x_0)$ because left and right limit of f coincide.[2]
Let now f and g be, respectively, the 2L-periodic extensions to $\mathbb{R}$ of x and x^2 from $[-L, L)$. Sketch a graph of these functions.

[2] This gives an answer to Exercise **2.b)**.

Look at the next page!

a) Are f and g well behaved in the sense specified above?

b) What are the points of discontinuity of f and g?

c) What are the mean values of the left and right limit of f in its points of discontinuity?

$$\frac{1}{2}\left(f^{+}(x_0)+f^{-}(x_0)\right) = \ ?$$

d) Does the Fourier series F of f converge to these values in these points? If the answer to **5.a)** is affirmative, then yes.
Verify it explicitely.

e) Prove that the Fourier series of g is

$$G(x) = \frac{L^2}{3} + \sum_{n=1}^{+\infty} (-1)^n \frac{4L^2}{\pi^2 n^2} \cos\left(\frac{n\pi}{L}x\right)$$

6. (Bonus exercise) With the same notations of the previous exercise.

a) Because g (the 2L-periodic extension of x^2) is well-behaved and continuous everywhere, its Fourier series G converge to it in every point. In particular

$$L^2 = g(L) = G(L).$$

Deduce from this equality the value of the Riemann Zeta function in 2

$$\zeta(2) := \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

b) Use this value to deduce that the limit of the square error for f computed in Exercise **4.** is zero.

c) Compute the square error for g, $E^*_N(g)$, and observe that the following are equivalent[3]

(i) $\lim_{N\to+\infty} E^*_N(g) = 0$

(ii) $\zeta(4) := \sum_{n=1}^{+\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

Due by: Thursday 1 / Friday 2 November 2018.

[3]in fact, they are (both) true.