D-MAVT
D-MATL

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Analysis III

Serie 7

What's needed: Fourier integral and Fourier transform.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be the characteristic function of the interval $[0, 1]$, that is

$$f(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find its Fourier integral representation

$$\int_0^{+\infty} \big(A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\big)\, d\omega$$

In which points does this integral actually coincide with f?
Explain why.

2. Same questions for the function

$$f(x) = \begin{cases} 1 + x, & -1 \leqslant x \leqslant 0 \\ 1 - x, & 0 \leqslant x \leqslant 1 \\ 0, & \text{otherwise.} \end{cases}$$

3. Let

$$f(x) = \begin{cases} \frac{\pi}{2}\sin(x), & 0 \leqslant |x| \leqslant \pi \\ 0, & \text{otherwise.} \end{cases}$$

Sketch the graph of this function, and prove that for every $x \in \mathbb{R}$

$$\int_0^{+\infty} \frac{\sin(\omega\pi)\sin(\omega x)}{1 - \omega^2}\, d\omega = f(x).$$

4. Find the Fourier transform $\widehat{f} = \mathcal{F}(f)$ of the following functions:

a) $f(x) = \begin{cases} e^{2ix}, & -1 \leqslant x \leqslant 1 \\ 0, & \text{otherwise} \end{cases}$

b) $f(x) = \begin{cases} x, & 0 \leqslant x \leqslant 1 \\ -x, & -1 \leqslant x \leqslant 0 \\ 0, & \text{otherwise.} \end{cases}$

5. **a)** Let $f : \mathbb{R} \to \mathbb{C}$ differentiable and such that

(i) $\lim_{|x| \to \infty} f(x) = 0$

(ii) f, f' are absolutely integrable.

Prove that

$$\widehat{(f')} = i\omega\widehat{f}.$$

b) Let $f : \mathbb{R} \to \mathbb{C}$ differentiable and such that

(i) f is absolutely integrable.

(ii) xf is also absolutely integrable.

With this conditions is legitimate to differentiate under the integral sign. Prove that the derivative of the Fourier transform is

$$\left(\widehat{f}\right)' = -i\widehat{(xf)}$$

6. **(Bonus exercise)** Consider the Gaussian function

$$f(x) = e^{-ax^2}, \qquad a > 0$$

and denote by $g := \widehat{f}$ its Fourier transform.

(i) Prove that, *for this particular choice of f,*

$$\widehat{(xf)} = -\frac{i\omega}{2a}\widehat{f} \left(= -\frac{i\omega}{2a} g \right).$$

(ii) Deduce from the previous point and Exercise **5.b)** that g satisfies the following differential equation

$$g' + \frac{\omega}{2a} g = 0.$$

(iii) Use the result of Exercise **5.a)** of Serie 2 to deduce that

$$g(0) = \frac{1}{\sqrt{2a}}.$$

(iv) From the previous points we get that g is the solution of the following IVP

$$\begin{cases} g' + \frac{\omega}{2a} g = 0 \\ g(0) = \frac{1}{\sqrt{2a}}. \end{cases}$$

Solve it, to deduce that the Fourier transform of the Gaussian function is

$$g(\omega) = \frac{1}{\sqrt{2a}} e^{-\frac{\omega^2}{4a}}.$$

Due by: Thursday 8 / Friday 9 November 2018.