D-MAVT
D-MATL

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Analysis III

Serie 8

What's needed: Introduction to PDEs, classification of 2nd order PDEs. Definition of wave equation, heat equation, Laplace/Poisson equation.

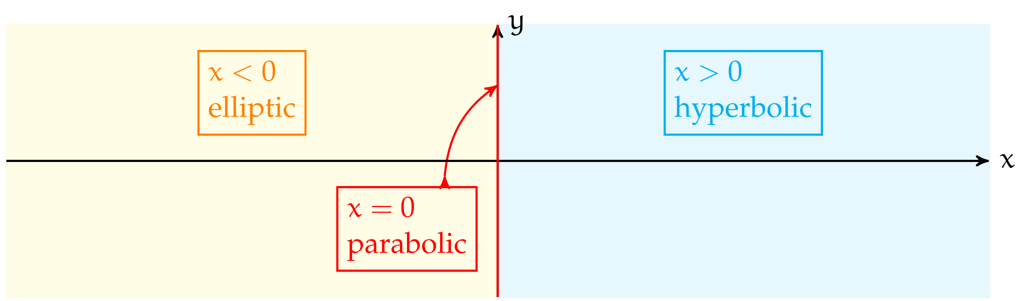
Recall: A 2nd order PDE is something of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$$

where the coefficients A, B, C may also be functions of x, y. To be precise, we say that the PDE is, respectively, hyperbolic, parabolic or elliptic, if the (minus) discriminant $AC - B^2$ is, respectively, always smaller, equal, or greater than zero. When the sign changes in different regions of the plane (x, y), the equation is called *of mixed type*.
For example the Euler-Tricomi equation

$$u_{xx} - xu_{yy} = 0$$

is of mixed type, and it is hyperbolic in the half plane $x > 0$, elliptic in the other half plane $x < 0$, and parabolic on the line $x = 0$.



1. Consider the following PDEs - in what follows, $u = u(x, y)$ is a function of two variables.

$$u_{xx} + 2u_{xy} + u_{yy} + 3u_x + xu = 0, \tag{1}$$
$$u_{xx} + 2u_{xy} + 2u_{yy} + u_y = 0, \tag{2}$$
$$u_{xx} + 8u_{xy} + 2u_{yy} + e^x u_x = 0, \tag{3}$$
$$yu_{xx} + 2xu_{xy} + u_{yy} - u_y = 0, \tag{4}$$
$$(x+1)u_{xx} + 2yu_{xy} + x^2 u_{yy} = 0. \tag{5}$$

Which of this is hyperbolic? Parabolic? Elliptic? Of mixed type?
In the last case, try to understand in which region of the plane (x, y) they are hyperbolic, parabolic or elliptic[1].

2. Consider the following functions.

a) $u(t, x) = e^{-100t} \cos(2x)$

b) $u(t, x) = \sin(2x) \cos(8t)$

c) $u(t, x) = e^{-36t} \sin(3x)$

Which PDE between the heat equation, $u_t = c^2 u_{xx}$, and the wave equation, $u_{tt} = c^2 u_{xx}$, does each of these solve? Write down also which is the constant c in each case.

3. Find the general solution $u = u(x, y)$ for the following PDEs:

a) $u_y + 2yu = 0$

b) $u_{yy} = 4xu_y$.

Due by: Thursday 15 / Friday 16 November 2018.

[1] you can plot the curve $\{AC - B^2 = 0\}$ on - say - Wolfram|Alpha to understand its shape.