D-MAVT
D-MATL

Analysis III

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Solutions Serie 10

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1. Consider the following PDE:

$$u_{xx} - 4u_{xy} + 3u_{yy} = 0.$$

Determine its type (hyperbolic, parabolic or elliptic), bring it in normal form with an opportune change of coordinates, and give all possible solutions.

Solution:
The PDE is in the form $Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0$, with $A = 1$, $B = -2$ and $C = 3$. It is hyperbolic because

$$AC - B^2 = 3 - 4 = -1 < 0.$$

The characteristic equation $A(y')^2 - 2By' + C = 0$ is

$$(y')^2 + 4y' + 3 = (y' + 1)(y' + 3) = 0,$$

which has solutions $y' = -1$ and $y' = -3$, that is

$$\begin{cases} y = -x + c_1 \\ y = -3x + c_2 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \xi(x,y) = x + y = c_1 \\ \zeta(x,y) = 3x + y = c_2 \end{cases} \quad (\ \xi(x,y), \zeta(x,y) \text{ are the characteristics })$$

The change of coordinates we have to perform is then given by[1]

$$\begin{aligned} v = \xi(x,y) = x + y \quad &\rightsquigarrow \quad v_x = 1,\ v_y = 1 \\ w = \zeta(x,y) = 3x + y \quad &\rightsquigarrow \quad w_x = 3,\ w_y = 1, \end{aligned}$$

from which we can compute the derivatives in this new coordinates

$$\begin{aligned} u_x &= u_v v_x + u_w w_x \\ &= u_v + 3u_w \\ u_{xx} &= u_{vv}v_x + 3u_{wv}v_x + u_{vw}w_x + 3u_{ww}w_x \\ &= u_{vv} + 6u_{vw} + 9u_{ww} \\ u_{xy} &= u_{vv}v_y + 3u_{wv}v_y + u_{vw}w_y + 3u_{ww}w_y \\ &= u_{vv} + 4u_{vw} + 3u_{ww} \\ u_y &= u_v v_y + u_w w_y \\ &= u_v + u_w \\ u_{yy} &= u_{vv}v_y + u_{wv}v_y + u_{vw}w_y + u_{ww}w_y \\ &= u_{vv} + 2u_{vw} + u_{ww}. \end{aligned}$$

[1] Recall: because the equation is hyperbolic we make the following change of coordinates: $(v,w) = (\xi,\zeta)$. If the equation was parabolic then we would have needed to use $(v,w) = (x,\zeta)$, and if it was elliptic $(v,w) = \big((\xi+\zeta)/2, (\xi-\zeta)/(2i)\big)$.

So, as expected, substituting in the equation we get the normal form $u_{vw} = 0$. From which all possible solutions are of the form

$$u(x, y) = \varphi(v) + \psi(w) = \varphi(x + y) + \psi(3x + y),$$

where φ and ψ are any couple of twice continuously-differentiable functions of one variable.

2. Let $u(x,t)$ be the solution of the following problem (1-dimensional wave equation on the line).

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x), & x \in \mathbb{R} \\ u_t(x,0) = 0, & x \in \mathbb{R} \end{cases}$$

where

$$f(x) = \begin{cases} e^{\frac{x^2}{x^2-1}}, & |x| < 1 \\ 0, & \text{otherwise.} \end{cases}$$

a) Sketch a graph of $f(x)$, which is the solution at the initial time.

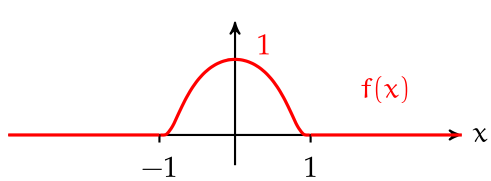
Solution:
Here's some hints on how to plot this function by yourself in the interval $(-1, 1)$. First of all you can observe that the function is always positive and never zero (because it's the exponential of something). Moreover we have

$$\lim_{|x|\to 1^-} \frac{x^2}{x^2-1} = -\infty \qquad \Longrightarrow \qquad \lim_{|x|\to 1^-} e^{\frac{x^2}{x^2-1}} = \lim_{t\to -\infty} e^t = 0,$$

so that the function joints continuously with the definition of $f(x)$ outside the interval [it's quite easy to see that the function is actually smooth - that is, it has all derivatives - but it's not relevant in our problem.]
The function is clearly even, so the graph is symmetric with respect the y-axis. One last observation that can help plotting the function quite accurately is either that it is decreasing from 0 to 1 (via derivative), or observing that actually

$$e^{\frac{x^2}{x^2-1}} = e^{\frac{x^2-1+1}{x^2-1}} = e^{\left(1+\frac{1}{x^2-1}\right)} = e \cdot e^{\frac{1}{x^2-1}}$$

and the latter is easier to understand. To summarize:



b) Sketch a graph of the solution at the time $t = 2, u(x, 2)$.

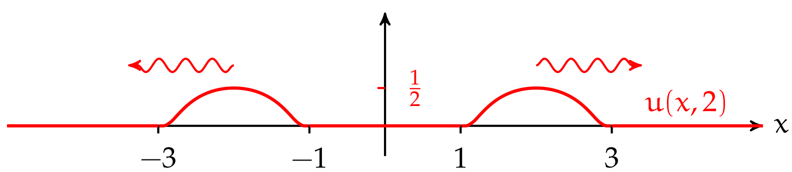
Solution:
D'Alembert solution of the wave equation is

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct))\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

which in this case simplifies to

$$u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t))\right).$$

This means that the initial 'wave' $f(x)$ is split in half and generates two waves, of its same shape (but each of half-height) going in opposite directions.



Remark: The wave equation has been discovered by d'Alembert in 1746, as the equation of a vibrating string. As the name suggests, this equation describes the behaviour of many waves: water waves, sound waves, light waves, seismic waves, and so on.
It is perhaps more enlightning to think about it in these terms, for example the Exercise you just solved describes quite well the shape of the waves you create if you let a stone fall in a lake (you should look at the surface of the lake from a side, so you see just a 1-dimensional profile). You can notice a little discrepancy with reality, because in reality usually there is not just one perfect block of two waves going in opposite directions, but also some other turbolences. This is in part due to the fact that actually the initial speed $g(x)$ is not zero. Adding this contribute to the solution would lead to an even more accurate picture.

c) Prove that, for each fixed $x \in \mathbb{R}$:

$$\lim_{t\to+\infty} u(x,t) = 0.$$

Give an explanation why this is always true if we start from

$$\begin{cases} f(x) \text{ such that: } \lim\limits_{|x|\to+\infty} f(x) = 0, \\ g(x) = 0. \end{cases}$$

Solution:
We already noted that in this case

$$u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t))\right).$$

For each fixed $x \in \mathbb{R}$ if we let t going to $+\infty$, the points $x \pm t$ will go, respectively, to $\pm\infty$. Therefore

$$\lim_{t\to+\infty} u(x,t) = \frac{1}{2}\left(\lim_{t\to+\infty} f(x+t) + \lim_{t\to+\infty} f(x-t)\right) = \frac{1}{2}\left(\lim_{s\to+\infty} f(s) + \lim_{s\to-\infty} f(s)\right) = 0.$$

Remark: More generally, for any wave equation with initial conditions such that

$$\begin{cases} \exists \lim\limits_{x\to\pm\infty} f(x) =: f(\pm\infty), \\ g(x) \text{ is integrable over all } \mathbb{R}, \end{cases}$$

we have

$$\lim_{t\to+\infty} u(x,t) = \frac{1}{2}\left(f(+\infty) + f(-\infty)\right) + \frac{1}{2c}\int_{-\infty}^{+\infty} g(s)\,ds.$$

3. Let $u(x,t)$ be the solution of the problem

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) = \begin{cases} 1, & |x| \leqslant 1 \\ 0, & |x| > 1 \end{cases}, & x \in \mathbb{R} \\ u_t(x,0) = g(x) = \begin{cases} 1, & |x| \leqslant 1 \\ 0, & |x| > 1 \end{cases}, & x \in \mathbb{R} \end{cases}$$

a) Find the values $u(0, \frac{1}{2})$ and $u(\frac{3}{2}, \frac{1}{2})$.

Solution:
We use d'Alembert formula to obtain

$$u(0, \tfrac{1}{2}) = \frac{1}{2}\left(f\left(\tfrac{1}{2}\right) + f\left(-\tfrac{1}{2}\right)\right) + \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} g(s)\,ds = \frac{1}{2}(1+1) + \frac{1}{2}\cdot 1 = \frac{3}{2}.$$

$$u(\tfrac{3}{2}, \tfrac{1}{2}) = \frac{1}{2}\left(f(2) + f(1)\right) + \frac{1}{2}\int_{1}^{2} g(s)\,ds = \frac{1}{2}(0+1) = \frac{1}{2}.$$

b) Find, for each fixed $x \in \mathbb{R}$, the limit

$$\lim_{t\to+\infty} u(x,t).$$

Solution:
We already observed in the previous exercise how to compute this limit. With the same notations, we substitute our initial conditions here to get, for each $x \in \mathbb{R}$:

$$\lim_{t\to+\infty} u(x,t) = \frac{1}{2}\left(f(+\infty) + f(-\infty)\right) + \frac{1}{2c}\int_{-\infty}^{+\infty} g(s)\,ds = \frac{1}{2}(0+0) + \frac{1}{2}\int_{-1}^{1} 1\,ds = 1.$$

Important remark: The limit we have computed in the previous Exercises $\lim\limits_{t\to+\infty} u(x,t)$ was independent from the $x \in \mathbb{R}$ chosen. Indeed, as said before, this was because f and g were well-behaved, in the sense that f had limits for $x \to \pm\infty$ and g was integrable, so that one can compute the limit of

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} g(s)\,ds$$

by computing the sum of the limits of all summands. But *pay attention*: this is not always the case!
Easy example in which this doesn't work: just take $c = 1$ and $f(x) = \sin(x)$, while $g(x) = 0$. Then clearly f doesn't have limits at infinity, and $u(x,t)$ will have limit for some x, but not for others:

$$\begin{cases} \lim\limits_{t\to+\infty} u(0,t) = \lim\limits_{t\to+\infty} \frac{1}{2}(\sin(t) - \sin(t)) = \lim\limits_{t\to+\infty} 0 = 0, \\ \lim\limits_{t\to+\infty} u(1,t) = \lim\limits_{t\to+\infty} \frac{1}{2}(\sin(1+t) + \sin(1-t)) = \lim\limits_{t\to+\infty} \sin(1)\cos(t) \quad \rightsquigarrow \quad \text{doesn't exist!} \end{cases}$$

More generally for each x and t

$$u(x,t) = \frac{1}{2}(\sin(x+t) + \sin(x-t)) = \cos(t)\sin(x)$$

and it will have limit for $t \to +\infty$ if and only if $x = k\pi$ with $k \in \mathbb{Z}$.