D-MAVT
D-MATL

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Analysis III

Solutions Serie 11

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1. Find, via Fourier series, the solution of the 1-dimensional heat equation with the following initial condition:

$$\begin{cases} u_t = 4\,u_{xx}, & \\ u(0,t) = u(1,t) = 0, & t \geqslant 0 \\ u(x,0) = f(x), & 0 \leqslant x \leqslant 1 \end{cases}$$

where

$$f(x) = \sin(\pi x) + \sin(5\pi x) + \sin(10\pi x).$$

Solution:
The parameters are $L = 1$ and thermal diffusivity $c^2 = 4$. So

$$\lambda_n^2 = \left(\frac{cn\pi}{L}\right)^2 = \frac{c^2n^2\pi^2}{L^2} = 4n^2\pi^2$$

and the solution of the heat equation via Fourier series will be

$$u(x,t) = \sum_{n=1}^{+\infty} B_n \sin(n\pi x) e^{-\lambda_n^2 t}$$

where the coefficients B_n are determined by the initial condition

$$f(x) = u(x,0) = \sum_{n=1}^{+\infty} B_n \sin(n\pi x).$$

This case is particularly easy because $f(x)$ is already expressed as a linear combination of these functions and there's no need to compute any integral to get

$$B_n = \begin{cases} 1, & n = 1,5,10 \\ 0, & \text{otherwise.} \end{cases}$$

Finally, the solution will be

$$u(x,t) = \sin(\pi x)e^{-4\pi^2 t} + \sin(5\pi x)e^{-100\pi^2 t} + \sin(10\pi x)e^{-400\pi^2 t}$$

2. An aluminium bar of length $L = 1(m)$ has thermal diffusivity of (around)[1]

$$c^2 = 0.0001\left(\frac{m^2}{\text{sec}}\right) = 10^{-4}\left(\frac{m^2}{\text{sec}}\right).$$

It has initial temperature given by $u(x,0) = f(x) = 100\sin(\pi x)$ (°C), and its ends are kept at a constant 0°C temperature. Find the first time t^* for which the whole bar will have temperature $\leqslant 30$°C.
In mathematical terms, solve

$$\begin{cases} u_t = 10^{-4}u_{xx}, \\ u(0,t) = u(1,t) = 0, & t \geqslant 0 \\ u(x,0) = 100\sin(\pi x), & 0 \leqslant x \leqslant 1. \end{cases}$$

and find the smallest t^* for which

$$\max_{x\in[0,1]} u(x,t^*) \leqslant 30.$$

Solution:
The parameters are length $L = 1$, thermal diffusivity $c^2 = 10^{-4}$ and consequently

$$\lambda_n^2 = \frac{c^2 n^2 \pi^2}{L^2} = 10^{-4}n^2\pi^2.$$

The solution is

$$u(x,t) = \sum_{n=1}^{+\infty} B_n \sin(n\pi x)e^{-\lambda_n^2 t}$$

and

$$f(x) = u(x,0) = \sum_{n=1}^{+\infty} B_n \sin(n\pi x)$$

so that the only nontrivial coefficients will be $B_1 = 100$. The solution is explicitely given by

$$u(x,t) = 100\sin(\pi x)e^{-10^{-4}\pi^2 t}.$$

For each fixed time $t \geqslant 0$, it is a multiple of $\sin(\pi x)$, therefore its maximum will be reached in $x = 1/2$ with value

$$M_t := \max_{x\in[0,1]} u(x,t) = u\left(\frac{1}{2}, t\right) = 100\sin\left(\frac{\pi}{2}\right)e^{-10^{-4}\pi^2 t} = 100e^{-10^{-4}\pi^2 t}.$$

This is a decreasing function of t, so that the required value t^* for which the bar will have temperature $\leqslant 30$°C is given by imposing

$$M_{t^*} = 30 \quad \Leftrightarrow \quad 100e^{-10^{-4}\pi^2 t^*} = 30 \quad \Leftrightarrow \quad t^* = \frac{10^4}{\pi^2}\ln\left(\frac{10}{3}\right)$$

$$\left(\approx 1219.88 \text{ sec} = 20 \text{ min } 19.88 \text{ sec}\right)$$

[1] we are approximating the standard value which would be $c^2 \approx 0.000097\text{m}^2/\text{sec}$ to make computations easier.

Look at the next page!

3. Solve the following Laplace equation (steady heat equation) on the rectangle

$$R = \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 2\},$$

$$\begin{cases} \Delta u = 0, & (x,y) \in R \\ u(0,y) = u(1,y) = 0, & 0 \leqslant y \leqslant 2 \\ u(x,0) = 0, & 0 \leqslant x \leqslant 1 \\ u(x,2) = f(x), & 0 \leqslant x \leqslant 1 \end{cases}$$

where

$$f(x) = x(1-x).$$

Solution:

The solution of the Dirichlet problem with this particular boundary conditions (u nontrivial only on the upper orizontal segment of the rectangle) has been given in the Lecture notes:

$$u(x,y) = \sum_{n=1}^{+\infty} A_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right),$$

where

$$A_n = \frac{2}{a \sinh\left(\frac{n\pi}{a}b\right)} \int_0^a f(x) \sin\left(\frac{n\pi}{a}x\right)\, dx$$

and the measures of the rectangle are, in this case, $a = 1$ and $b = 2$. So we basically just need to compute, with a few integration by parts,

$$\int_0^1 f(x)\sin(n\pi x)\,dx = \int_0^1 x(1-x)\sin(n\pi x)\,dx =$$

$$= \cancel{-x(1-x)\frac{\cos(n\pi x)}{n\pi}\Bigg|_0^1} + \frac{1}{n\pi}\int_0^1 (1-2x)\cos(n\pi x)\,dx =$$

$$= \frac{1}{n\pi}\left(\cancel{(1-2x)\frac{\sin(n\pi x)}{n\pi}\Bigg|_0^1} + \frac{2}{n\pi}\int_0^1 \sin(n\pi x)\,dx \right) =$$

$$= \frac{1}{n\pi}\left(-\frac{2}{n^2\pi^2}\cos(n\pi x)\Bigg|_0^1 \right) = -\frac{2}{n^3\pi^3}(\cos(n\pi) - 1) =$$

$$= \frac{2}{n^3\pi^3}(1-(-1)^n) = \begin{cases} \frac{4}{(2k+1)^3\pi^3}, & n = 2k+1 \\ 0, & \text{otherwise.} \end{cases}$$

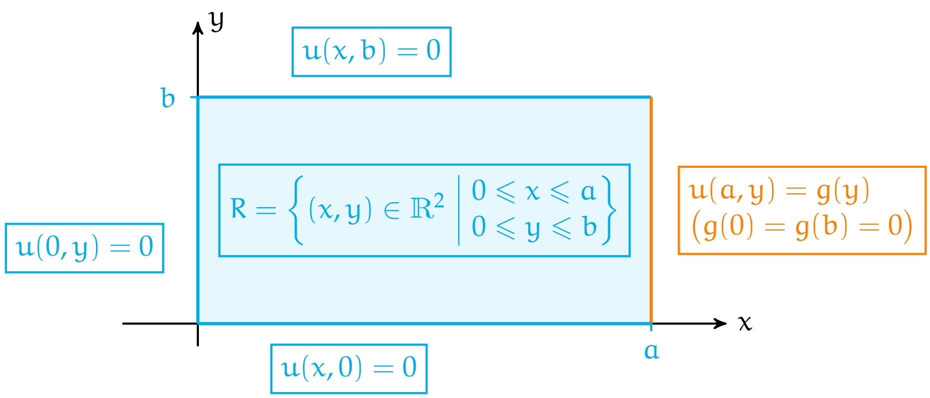
We can now find the coefficients

$$A_n = \frac{2}{\sinh(2n\pi)} \int_0^1 f(x)\sin(n\pi x)\,dx = \begin{cases} \frac{8}{\sinh(2(2k+1)\pi)(2k+1)^3\pi^3}, & n = 2k+1 \\ 0, & \text{otherwise.} \end{cases}$$

and the solution will be

$$u(x,y) = 8\sum_{k=0}^{+\infty} \frac{\sin\left((2k+1)\pi x\right)\sinh\left((2k+1)\pi y\right)}{\sinh(2(2k+1)\pi)(2k+1)^3\pi^3}.$$

4. Adapt the method used to solve the previous Laplace equation in the case in which the only nontrivial initial boundary condition is on the right vertical segment of the rectangle



$$\begin{cases} \Delta u = 0, & (x,y) \in R \\ u(x,0) = u(x,b) = 0, & 0 \leqslant x \leqslant a \\ u(0,y) = 0, & 0 \leqslant y \leqslant b \\ u(a,y) = g(y), & 0 \leqslant y \leqslant b \end{cases}$$

where $g(y)$ is any function with prescribed boundary conditions

$$g(0) = g(b) = 0.$$

Solution:
We just have to make a few changes from the way the equation was solved in the lecture notes. To solve the differential equation $\Delta u = 0$ by separation of variables

$$u(x,y) = F(x)G(y)$$

we still have to impose for some $k \in \mathbb{R}$:

$$\begin{cases} F'' = -kF \\ G'' = kG. \end{cases}$$

We first impose the boundary conditions $u(x,0) = u(x,b) = 0$, which translate into $G(0) = G(b) = 0$. To have nontrivial solutions, we must have $k < 0$. With this condition we solve

$$\begin{cases} G'' = kG \\ G(0) = G(b) = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} G(y) = A\cos\left(\sqrt{-k}y\right) + B\sin\left(\sqrt{-k}y\right) \\ G(0) = G(b) = 0 \end{cases} \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \begin{cases} (G(0)=0) & A=0 \\ (G(b)=0) & \sqrt{-k}b = n\pi \ (n \in \mathbb{Z}_{\geqslant 1}) \end{cases} \quad \rightsquigarrow \quad G_n(y) = B_n \sin\left(\frac{n\pi}{b}y\right), \quad n \geqslant 1$$

For these admissible values we found

$$\sqrt{-k} = \frac{n\pi}{b} \quad \rightsquigarrow \quad k = -\left(\frac{n\pi}{b}\right)^2$$

we have solutions of the other differential equation $F'' = -kF$

$$F_n(x) = A_n^* e^{\frac{n\pi}{b}x} + B_n^* e^{-\frac{n\pi}{b}x}$$

and imposing the boundary condition $u(0,y) = 0$ we have $F_n(0) = 0$, that is

$$F_n(x) = 2A_n^* \sinh\left(\frac{n\pi}{b}x\right).$$

Renaming the product of the constants $A_n := B_n \cdot 2A_n^*$ we get

$$u_n(x,y) = F_n(x)G_n(y) = A_n \sinh\left(\frac{n\pi}{b}x\right)\sin\left(\frac{n\pi}{b}y\right),$$

and by the superposition principle

$$u(x,y) = \sum_{n=1}^{+\infty} u_n(x,y) = \sum_{n=1}^{+\infty} A_n \sinh\left(\frac{n\pi}{b}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

is also a solution. We now only have to impose the last boundary condition $u(a,y) = g(y)$ which translates into

$$g(y) = \sum_{n=1}^{+\infty} \left[A_n \sinh\left(\frac{n\pi}{b}a\right)\right]\sin\left(\frac{n\pi}{b}y\right)$$

so that the expressions in the square brackets must be the coefficients of the odd, 2b-periodic extension of $g(y)$, or equivalently

$$A_n = \frac{2}{b\sinh\left(\frac{n\pi}{b}a\right)}\int_0^b g(y)\sin\left(\frac{n\pi}{b}y\right)\, dy.$$