D-MAVT
D-MATL

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Autumn 2018

Analysis III

Solutions Serie 12

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1. (Old exercise)

a) Using the method of separation of variables, find a Fourier series solution for the following problem:

$$\begin{cases} u_t = c^2 u_{xx}, & 0 \leqslant x \leqslant L, t \geqslant 0 \\ u(0,t) = u(L,t) = 0, & t \geqslant 0 \\ u_x(x,0) = h(x), & 0 \leqslant x \leqslant L \end{cases}$$

where $h(x)$ is any (differentiable) function such that

$$\int_0^L h(x)\,dx = 0.$$

Why do we need to require this condition?

Solution:
The general solution via separation of variables of the heat equation with zero boundary conditions is

$$u(x,t) = \sum_{n=1}^{+\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}, \qquad \lambda_n^2 = \left(\frac{cn\pi}{L}\right)^2.$$

The difference from the problem solved in the Lecture notes is that we need to find the coefficients B_n by imposing this other initial condition.
<u>1$^{\text{st}}$ method:</u> deriving (in the variable x) term by term the previous expression we obtain

$$u_x(x,t) = \sum_{n=1}^{+\infty} B_n \frac{n\pi}{L} \cos\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t},$$

and we have to impose at the time $t = 0$:

$$h(x) = \sum_{n=1}^{+\infty} B_n \frac{n\pi}{L} \cos\left(\frac{n\pi}{L}x\right).$$

The right-hand side looks very much like the Fourier series of a 2L-periodic, even function. The only possible difference is that there should also be a term for $n = 0$, while here there's not. But the term corresponding to $n = 0$ is given by the integral of the function, so it is zero if and only if this integral vanishes. Our function $h(x)$ has exactly this property by hypothesis, and therefore we can conclude that

$$B_n \frac{n\pi}{L} \underset{\substack{\text{Fourier coefficients of the 2L-periodic,} \\ \text{even extension of } h(x) \text{ from } [0,L]}}{=} \frac{2}{L} \int_0^L h(x) \cos\left(\frac{n\pi}{L} x\right)\, dx,$$

or equivalently

$$B_n = \frac{2}{n\pi} \int_0^L h(x) \cos\left(\frac{n\pi}{L} x\right)\, dx.$$

<u>2nd method:</u> imposing the initial condition $u_x(x, 0) = h(x)$ is equivalent to impose

$$u(x, 0) = f(x), \qquad \text{where } f(x) := \int_0^x h(s)\, ds$$

$$\big(\rightsquigarrow \quad f'(x) = h(x) \quad \& \quad f(0) = f(L) = 0 \big)$$

For this we can use the usual formula and conclude that

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right)\, dx.$$

We must now reformulate this condition in terms of $h(x)$, because this was the initial datum of the problem. To do it, it's enough to integrate by parts and obtain again

$$\begin{aligned}
B_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right)\, dx = \frac{2}{\cancel{L}} \int_0^L f(x) \left(-\frac{\cancel{L}}{n\pi} \cos\left(\frac{n\pi}{L} x\right) \right)' dx = \\
&= -\frac{2}{n\pi} \int_0^L f(x) \left(\cos\left(\frac{n\pi}{L} x\right) \right)' dx = \\
&= -\frac{2}{n\pi} \left[\cancel{f(x) \cos\left(\frac{n\pi}{L} x\right) \Big|_0^L} - \int_0^L h(x) \cos\left(\frac{n\pi}{L} x\right)\, dx \right] = \\
&= \frac{2}{n\pi} \int_0^L h(x) \cos\left(\frac{n\pi}{L} x\right)\, dx.
\end{aligned}$$

<u>Why do we need to require that condition on $h(x)$?</u>
It's clear from the solution where we used the condition that the integral of $h(x)$ vanishes. There is, however, even before trying to solve the problem, a

more conceptual reason why we need to impose this condition. Being $h(x)$ the derivative of $u(x,0)$ we must have

$$\int_0^L h(x)\,dx = \int u_x(x,0)\,dx = u(L,0) - u(0,0) = 0.$$

In other words the vanishing condition of the integral is necessary for $h(x)$ to be compatible with the other data of the problem. As will be explained at the end of the course, this is a matter of well/ill-posed problem (if the conditions are not compatible, there is no solution).

b) Find the general solution for the following problem:

$$\begin{cases} u_t = c^2 u_{xx}, & 0 \leqslant x \leqslant L, t \geqslant 0 \\ u(0,t) = a, & t \geqslant 0 \\ u(L,t) = b, & t \geqslant 0 \\ u(x,0) = f(x), & 0 \leqslant x \leqslant L \end{cases}$$

where $a, b \in \mathbb{R}$ are arbitrary constants, and $f(x)$ is any (twice differentiable) function such that $f(0) = a, f(L) = b$.
Compute, for each fixed $0 \leqslant x \leqslant L$, the asymptotic limit

$$\lim_{t \to +\infty} u(x,t).$$

Solution:
We know how to solve the problem with zero boundary values. So we want to modify u by an opportune function in such a way that we are brought to this situation.
We would like to change only the boundary condition and not the equation, so the easiest way to do it is using a function (independent from time) whose 2nd derivative in the variable x vanishes.
We are therefore looking for the linear function $l(x)$ which assumes, respectively, the values a, b in the points $0, L$. Which is

$$l(x) = a + \frac{(b-a)}{L} x.$$

Performing the substitution $u = v + l$, u will solve the initial problem if and only if v will solve

$$\begin{cases} v_t = c^2 v_{xx}, & 0 \leqslant x \leqslant L, t \geqslant 0 \\ v(0,t) = 0, & t \geqslant 0 \\ v(L,t) = 0, & t \geqslant 0 \\ v(x,0) = \tilde{f}(x), & 0 \leqslant x \leqslant L \end{cases}$$

$$\tilde{f}(x) = f(x) - l(x), \qquad \left(\tilde{f}(0) = \tilde{f}(L) = 0\right)$$

and we know that the solution to this problem is

$$v(x,t) = \sum_{n=1}^{+\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t},$$

with

$$B_n = \frac{2}{L}\int_0^L \tilde{f}(x) \sin\left(\frac{n\pi}{L}x\right)\, dx.$$

Performing the integration of the $l(x)$ part we can be even more explicit, and express B_n in terms of the initial datum of the problem $f(x)$

$$B_n = \cdots = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right)\, dx + \frac{2}{\pi n}\left((-1)^n b - a\right).$$

With these coefficients, the final solution will be

$$u(x,t) = \underbrace{\sum_{n=1}^{+\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}}_{v(x,t)} + \underbrace{a + \frac{(b-a)}{L}x}_{l(x)}.$$

To compute the asymptotic limit, let's first observe that the asymptotic limit of the solution of the heat equation with zero boundary conditions is zero. In fact, one can exchange limit and infinit sum[1] to obtain

$$\lim_{t\to+\infty} v(x,t) = \lim_{t\to+\infty} \sum_{n=1}^{+\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t} =$$

$$= \sum_{n=1}^{+\infty} \lim_{t\to+\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t} = 0.$$

But then the asymptotic limit of our solution will be just the linear function

$$\lim_{t\to+\infty} u(x,t) = \cancel{\lim_{t\to+\infty} v(x,t)} + l(x) \left(= a + \frac{(b-a)}{L}x\right).$$

Remark: The asymptotic limit of this problem (heat equation + boundary and initial conditions given) is given by the linear function $l(x)$, which is the solution of the steady heat equation with same data:

$$\begin{cases} l_{xx} = 0, & 0 \leqslant x \leqslant L \\ l(0) = a, \\ l(L) = b. \end{cases}$$

[1] because of Lebesgue's dominated convergence theorem

Look at the next page!

2. Consider the following 1-dimensional heat equation, with initial temperature given by a gaussian distribution

$$\begin{cases} u_t = c^2 u_{xx}, & x \in \mathbb{R}, t \geqslant 0 \\ u(x,0) = e^{-x^2}, & x \in \mathbb{R} \end{cases}$$

a) Find an explicit formula for the solution $u(x,t)$ (explicit = no unsolved integrals or other implicit computations left).

Solution:
Let's Fourier transform (with respect to the variable x) the problem. We are going to denote by $U = \mathcal{F}(u)$, or more precisely, for each $t \geqslant 0$,

$$U(t) = \mathcal{F}\big(u(\cdot,t)\big).$$

Using Exercise **6.** in Serie 7 for the Fourier transform of the gaussian function and the Exercise **5.a)** in Serie 7 for the Fourier transform of a derivative, we get

$$\begin{cases} \dot{U}(t) \overset{\textbf{5.a)}}{=} -c^2\omega^2 U(t) \\ U(0) = \mathcal{F}\left(e^{-x^2}\right) \overset{\textbf{6.}}{=} \frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{4}} \end{cases} \quad \rightsquigarrow \quad U(t) = \frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{4}} e^{-c^2\omega^2 t} = \frac{1}{\sqrt{2}} e^{-\frac{(1+4c^2t)}{4}\omega^2}$$

which is still a gaussian function. We can find its inverse by trying to match the coefficients and we will find

$$u(x,t) = \mathcal{F}^{-1}\left(U(t)\right) = \frac{1}{\sqrt{1+4c^2t}} e^{-\frac{x^2}{1+4c^2t}}.$$

b) Say if the following equalities are true or false.

(i) $u\left(0, \frac{1}{4c^2}\right) = \frac{\sqrt{2}}{2}$ ~~(T)~~ (F)

Solution:

$$u\left(0, \frac{1}{4c^2}\right) = \frac{1}{\sqrt{1+4c^2 \cdot \frac{1}{4c^2}}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

(ii) $u(1,1) = \frac{1}{\sqrt{1+4c^2}} e^{\frac{1}{1+4c^2}}$ (T) ~~(F)~~

Solution:

$$u(1,1) = \frac{1}{\sqrt{1+4c^2}} e^{-\frac{1}{1+4c^2}}.$$

(iii) $u(0,1) = \frac{1}{\sqrt{1+4c^2}}$ ~~(T)~~ (F)

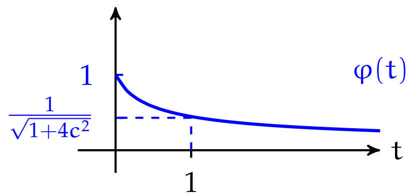
Solution:

$$u(0,1) = \frac{1}{\sqrt{1+4c^2}} e^0 = \frac{1}{\sqrt{1+4c^2}}.$$

c) Write explicitely the function $\varphi(t) := u(0,t)$ which describes, for $t \geqslant 0$, the evolution of the temperature in the point $x = 0$, and sketch a graph of it.

Solution:

$$\varphi(t) = u(0,t) = \frac{1}{\sqrt{1+4c^2t}}.$$



d) Find, for each each temperature $0 < \lambda < 1$, the only time $t = t(\lambda)$ for which the point at the origin $x = 0$ will have temperature equal to λ.

Solution:
We impose

$$\lambda = \varphi(t) = \frac{1}{\sqrt{1+4c^2t}} \quad \Leftrightarrow \quad \lambda^2 = \frac{1}{1+4c^2t} \quad \Leftrightarrow \quad t = \frac{1-\lambda^2}{4c^2\lambda^2}.$$

3. <u>(Bonus exercise - not treated in the lecture)</u>
An elastic membrane of squared shape of side length 1 m is let vibrating from the initial position described by the function $f(x,y)$ below, with initial speed zero. The material of which the membrane is composed is such that its vibrating waves will propagate with speed $c = 1\,\text{m/s}$.
In mathematical terms the profile of the membrane at the time t is described by the function $u(x,y,t)$ which is the solution of the following problem

$$R := \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leqslant x,y \leqslant 1\}$$

$$\begin{cases} u_{tt} = u_{xx} + u_{yy}, & (x,y) \in R, \quad t \geqslant 0 \\ u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0, & 0 \leqslant x,y \leqslant 1, \ t \geqslant 0 \\ u(x,y,0) = f(x,y) = \sin(\pi x)\sin(2\pi y), & (x,y) \in R \\ u_t(x,y,0) = 0. & (x,y) \in R \end{cases}$$

a) Find the solution $u(x,y,t)$.

Solution:
By separation of variables as explained in the Lecture notes, to solve the differential equation with zero boundary condition we are lead to a general solution of the form

$$u(x,y,t) = \sum_{m,n=1}^{+\infty} \left[B_{mn}\cos(\lambda_{mn}t) + B^*_{mn}\sin(\lambda_{mn}t)\right]\sin(m\pi x)\sin(n\pi y),$$

$$\lambda_{mn} = \pi\sqrt{m^2+n^2}.$$

We now have to impose the initial conditions on the position and velocity, which will lead to determine the coefficients B_{mn}, B^*_{mn}. In fact

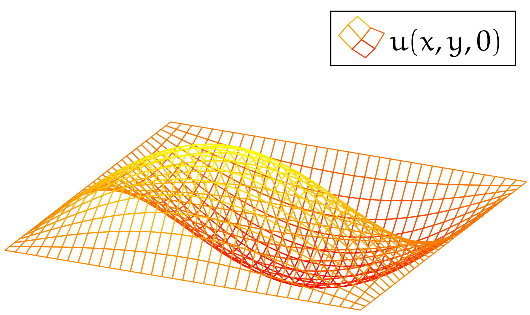
$$\begin{cases} u(x,y,0) = \sum\limits_{m,n=1}^{+\infty} B_{mn} \sin(m\pi x)\sin(n\pi y) = f(x,y) \\ u_t(x,y,0) = \sum\limits_{m,n=1}^{+\infty} B^*_{mn}\lambda_{mn} \sin(m\pi x)\sin(n\pi y) = 0 \end{cases}$$

$$\rightsquigarrow \qquad \begin{cases} B_{12} = 1, \quad \& \quad B_{mn} = 0 \quad \text{otherwise} \\ B^*_{mn} = 0. \end{cases}$$

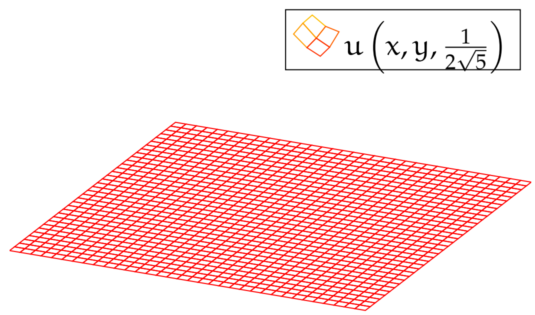
The corresponding value to compute is $\lambda_{12} = \pi\sqrt{1^2+2^2} = \sqrt{5}\pi$, and the solution is

$$u(x,y,t) = \cos\left(\sqrt{5}\pi t\right)\sin(\pi x)\sin(2\pi y).$$

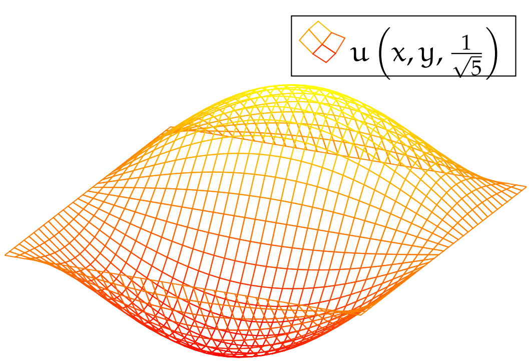
Initial configuration



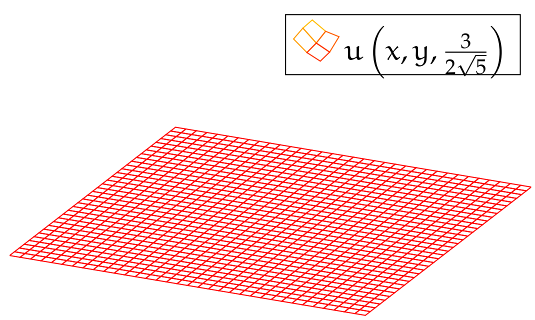
Evolution after around 0.22 sec



Evolution after around 0.44 sec



Evolution after around 0.66 sec



b) What is the height range in which the membrane vibrates? That is, find

$$\text{height}_{\text{min}} := \min_{\substack{(x,y)\in R\\ t\geqslant 0}} u(x,y,t), \qquad \text{height}_{\text{max}} := \max_{\substack{(x,y)\in R\\ t\geqslant 0}} u(x,y,t)$$

Solution:
The solution is product of functions which assume all possible values from -1 to 1. Therefore all these values will be assumed by $u(x,y,t)$ itself and the height range is

$$\text{height}_{\text{min}} = -1 \qquad \& \qquad \text{height}_{\text{max}} = 1.$$

c) Find the instants in which the membrane is completely flat. That is, find the $t^* \geqslant 0$ such that

$$u(x,y,t^*) = 0, \quad \forall (x,y) \in R.$$

Solution:

$$u(x,y,t^*) = 0,\ \forall(x,y)\in R \quad \Leftrightarrow \quad \cos\left(\sqrt{5}\pi t^*\right) = 0 \quad \Leftrightarrow \quad t^* = \frac{2k+1}{2\sqrt{5}},\ \ k \in \mathbb{N}.$$

d) Find the instants in which the membrane is momentaneously still. That is, find the $t^* \geqslant 0$ such that

$$u_t(x,y,t^*) = 0, \quad \forall (x,y) \in R.$$

Solution:
The speed is $u_t(x,y,t) = -\sqrt{5}\pi \sin\left(\sqrt{5}\pi t\right)\sin(\pi x)\sin(2\pi y)$. Therefore

$$u_t(x,y,t^*) = 0,\ \forall(x,y)\in R \quad \Leftrightarrow \quad \sin\left(\sqrt{5}\pi t^*\right) = 0 \quad \Leftrightarrow \quad t^* = \frac{k}{\sqrt{5}},\ \ k \in \mathbb{N}.$$

e) Is the vibration of the membrane periodic? If yes, what is the fundamental period?

Solution:
Again, this is a question regarding only the function $\cos\left(\sqrt{5}\pi t\right)$, which is - as explained in the Exercise **1.b)** of Serie 5 - periodic of fundamental period P given by

$$\sqrt{5}\pi P = 2\pi \quad \rightsquigarrow \quad P = \frac{2}{\sqrt{5}}.$$