Analysis III

Solutions Serie 14

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1. Let u the unique harmonic function on the disk of radius R which on the boundary is

$$u(x,y) = x^2y^2, \qquad (x,y) \in \partial D_R.$$

Answer, *without finding explicitely the function on the whole disk*, the following questions.

a) What's the value in the center of the disk $u(0,0) = ?$

Solution:
We can recover the value of the function in the center of the disk just from the boundary. In fact Poisson's integral formula tells us that for each point (r,θ) inside the disk

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)f(\phi)}{R^2 - 2rR\cos(\theta - \phi) + r^2}\, d\phi$$

In particular the value in the center computes the average of $f(\phi)$ on the boundary

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\, d\phi.$$

We rewrite our boundary condition in polar coordinates

$$x^2y^2 = (R\cos(\phi))^2(R\sin(\phi))^2 = R^4\cos^2(\phi)\sin^2(\phi),$$

and obtain

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} R^4\cos^2(\phi)\sin^2(\phi)\, d\phi = \frac{R^4}{2\pi}\int_0^{2\pi} \frac{\sin^2(2\phi)}{4}\, d\phi = \frac{R^4}{8\pi}\int_0^{2\pi} \sin^2(2\phi)\, d\phi.$$

This can be either computed with usual integration by parts or using this
Alternative method: observe that

$$\sin\left(x + \frac{\pi}{2}\right) = -\cos(x) \qquad \rightsquigarrow \qquad \sin^2\left(x + \frac{\pi}{2}\right) = \cos^2(x). \tag{1}$$

But then changing variables $\psi = \phi - \pi/4$, and using the 2π-periodicity of all these functions we get

$$\int_0^{2\pi} \sin^2(2\phi)\, d\phi \overset{\substack{\text{change}\\ \text{of}\\ \text{variables}}}{=} \int_{-\frac{\pi}{4}}^{2\pi-\frac{\pi}{4}} \sin^2\left(2\psi + \frac{\pi}{2}\right) d\psi \overset{(1)}{=} \int_{-\frac{\pi}{4}}^{2\pi-\frac{\pi}{4}} \cos^2(2\psi)\, d\psi \overset{\substack{2\pi\text{-}\\ \text{periodicity}}}{=}$$

$$= \int_0^{2\pi} \cos^2(2\psi)\, d\psi.$$

Finally observe that the sum of these two (equal) integral is

$$\int_0^{2\pi} \sin^2(2\phi)\, d\phi + \int_0^{2\pi} \cos^2(2\phi)\, d\phi = \int_0^{2\pi} \left(\sin^2(2\phi) + \cos^2(2\phi)\right) d\phi = \int_0^{2\pi} 1\, d\phi = 2\pi$$

and so

$$\rightsquigarrow \qquad \int_0^{2\pi} \sin^2(2\phi)\, d\phi = \int_0^{2\pi} \cos^2(2\phi)\, d\phi = \pi.$$

Coming back to the original question of the exericse

$$u(0,0) = \frac{R^4}{8\pi} \int_0^{2\pi} \sin^2(2\phi)\, d\phi = \frac{R^4}{8\cancel{\pi}} \cdot \cancel{\pi} = \frac{R^4}{8}.$$

b) What's the maximum of u and in which point(s) is it reached?

Solution:
By the maximum principle we know that the maximum is assumed on the boundary

$$\max_{(x,y)\in D_R} u(x,y) = \max_{(x,y)\in \partial D_R} u(x,y) = \max_{\theta\in[0,2\pi)} u(R,\theta) = \max_{\theta\in[0,2\pi)} \frac{R^4}{4} \sin^2(2\theta)$$

The maximum value for the square of the sine is assumed when

$$\sin^2(2\theta) = \pm 1 \quad \Leftrightarrow \quad \theta = \frac{\pi}{4}, \frac{3}{4}\pi, \frac{5}{4}\pi, \frac{7}{4}\pi.$$

So the maximum is reached in 4 points on the boundary and it's equal to

$$P_j = \left(R, \frac{2j+1}{4}\pi\right), \quad j = 0,1,2,3 \qquad \rightsquigarrow \qquad u(P_j) \equiv \frac{R^4}{4}$$

c) Same question for the minimum.

Look at the next page!

Solution:
As the maximum principle, there is also a minimum principle. Observe that a function u solves the Dirichlet problem

$$\begin{cases} \nabla^2 u = 0, & \text{in } D_R \\ u = f & \text{on } \partial D_R \end{cases}$$

if and only if the function $v := -u$ solves the Dirichlet problem

$$\begin{cases} \nabla^2 v = 0, & \text{in } D_R \\ v = -f. & \text{on } \partial D_R \end{cases}$$

Therefore, using the maximum principle for v, and the observation that for any function h its maximum and minimum are linked by

$$\max h = -\min(-h) \quad \& \quad \min h = -\max(-h),$$

we get the minimum principle: also the minimum of a harmonic function is reached on the boundary.

$$\min_{D_R} u = -\max_{D_R}(-u) = -\max_{D_R} v = -\max_{\partial D_R} v = -\max_{\partial D_R}(-u) = \min_{\partial D_R} u.$$

So we need to find the minimum

$$\min_{D_R} u = \min_{\partial D_R} u = \min_{\theta\in[0,2\pi)} u(R,\theta) = \min_{\theta\in[0,2\pi)} \frac{R^4}{4}\sin^2(2\theta)$$

which is clearly reached in the following 4 points with value

$$Q_j = \left(R, \frac{j}{2}\pi\right), \quad j = 0,1,2,3 \qquad \rightsquigarrow \qquad u(Q_j) \equiv 0.$$

2. For each of the following problems, determine whether they admit a (at least one) solution or not.

a)

$$\begin{cases} \nabla^2 u = 0, & \text{in } D_R \\ u = x^a y^b, & \text{on } \partial D_R \\ u(0,0) = 0. \end{cases}$$

where $a, b \geqslant 0$ are integer numbers.

Solution:
The Dirichlet problem with boundary condition has a unique solution. The value in the center is determined by the average of the boundary function. In polar coordinates our function is

$$x^a y^b = (R\cos(\theta))^a (R\sin(\theta))^b = R^{a+b}\cos^a(\theta)\sin^b(\theta).$$

Therefore the problem above has solution if and only if

$$\frac{1}{2\pi}\int_0^{2\pi} R^{a+b}\cos^a(\theta)\sin^b(\theta)\,d\theta = 0.$$

Let's call, for each integers $a, b \geqslant 0$, the relevant part of this integral by $I_{a,b}$ and observe that, by 2π-periodicity, it is also equal to

$$I_{a,b} := \int_0^{2\pi}\cos^a(\theta)\sin^b(\theta)\,d\theta = \int_{-\pi}^{\pi}\cos^a(\theta)\sin^b(\theta)\,d\theta.$$

Observe that

$$\begin{cases}\cos(\theta) \text{ even}\\ \sin(\theta) \text{ odd}\end{cases} \quad\rightsquigarrow\quad \begin{cases}\cos^a(\theta) \ \text{even}, & \forall a\\ \sin^b(\theta) & \begin{cases}\text{even}, & b \text{ even}\\ \text{odd}, & b \text{ odd}\end{cases}\end{cases}$$

We can conclude that for b is odd, and any a, we integrate an odd function on $[-\pi,\pi]$ and therefore $I_{a,b} = 0$. We need to analyse the case $b = 2k$ even, for which the product is even and we have

$$I_{a,2k} = \int_{-\pi}^{\pi}\cos^a(\theta)\sin^{2k}(\theta)\,d\theta = 2\int_0^{\pi}\cos^a(\theta)\sin^{2k}(\theta)\,d\theta.$$

If we split the interval $[0,\pi]$ in half we can notice one last simmetry, the one with respect to the axis $x = \pi/2$, and splitting the integral in these two parts we get

$$\begin{cases}\cos(\pi - x) = -\cos(x)\\ \sin(\pi - x) = \sin(x)\end{cases} \quad\rightsquigarrow\quad \int_0^{\pi}\cos^a(\theta)\sin^{2k}(\theta)\,d\theta =$$

$$= \int_0^{\frac{\pi}{2}}\cos^a(\theta)\sin^{2k}(\theta)\,d\theta + \int_{\frac{\pi}{2}}^{\pi}\underbrace{\cos^a(\theta)\sin^{2k}(\theta)\,d\theta}_{\text{change of variables } \ \theta=\pi-x}$$

$$= \int_0^{\frac{\pi}{2}}\cos^a(\theta)\sin^{2k}(\theta)\,d\theta + (-1)^a\int_0^{\frac{\pi}{2}}\cos^a(x)\sin^{2k}(x)\,dx =$$

$$= \begin{cases}2\int_0^{\frac{\pi}{2}}\cos^a(\theta)\sin^{2k}(\theta)\,d\theta > 0, & a \ \text{even}\\ 0, & a \ \text{odd}.\end{cases}$$

The reason why the first integral is greater than zero is because the integrand is a positive function on that interval.
The conclusion is

$$\begin{array}{l}\text{the problem}\\ \text{above admits}\\ \text{a solution}\end{array} \quad\Leftrightarrow\quad I_{a,b} = 0 \quad\Leftrightarrow\quad \begin{cases}b \ \text{odd}, \ \forall a\\ b \ \text{even}, \ a \ \text{odd}.\end{cases}$$

b)

$$\begin{cases} \nabla^2 u = 0, & \text{in } D_R \\ u(R,\theta) = 3R\, e^{\frac{(\theta-\pi)^2}{\theta(\theta-2\pi)}}, & 0 \leqslant \theta \leqslant 2\pi \ \big(\text{ parametrising } \partial D_R\big) \\ \max\limits_{D_R} u = \pi \end{cases}$$

Solution:
By the maximum principle the maximum is reached on the boundary

$$\max_{D_R} u = \max_{\partial D_R} u = \max_{\theta\in[0,2\pi)} u(R,\theta) = 3R \max_{\theta\in[0,2\pi)} e^{\frac{(\theta-\pi)^2}{\theta(\theta-2\pi)}}.$$

The exponential is a strictly increasing function so we just need to find the maximum of the argument, that is

$$\max_{\theta\in[0,2\pi)} e^{\frac{(\theta-\pi)^2}{\theta(\theta-2\pi)}} = e^{\max\limits_{\theta\in[0,2\pi)} \frac{(\theta-\pi)^2}{\theta(\theta-2\pi)}}$$

Analysing this rational function

$$g(\theta) = \frac{(\theta-\pi)^2}{\theta(\theta-2\pi)}, \quad \theta \in [0, 2\pi)$$

we notice that the limit approaching 0 from the right (and 2π from the left) is $-\infty$, and that it is always strictly negative, apart from $\theta = \pi$, in which it's zero. Therefore

$$\max_{D_R} u = 3R \max_{\theta\in[0,2\pi)} e^{\frac{(\theta-\pi)^2}{\theta(\theta-2\pi)}} = 3Re^{\max\limits_{\theta\in[0,2\pi)} \frac{(\theta-\pi)^2}{\theta(\theta-2\pi)}} = 3Re^0 = 3R.$$

The conclusion is

$$\begin{array}{l}\text{the problem}\\ \text{above admits}\\ \text{a solution}\end{array} \quad \Leftrightarrow \quad 3R = \pi \quad \Leftrightarrow \quad R = \frac{\pi}{3}$$

c)

$$\begin{cases} \nabla^2 u = 0, & \text{in } D_R \\ u(R,\theta) = \sin^9(\theta), & 0 \leqslant \theta \leqslant 2\pi \ \big(\text{ parametrising } \partial D_R\big) \\ u + 1 \geqslant 0, & \text{in } D_R \end{cases}$$

Solution:
By the minimum principle (explained above), the minimum will be on the boundary, therefore

$$\min_{D_R} u = \min_{\theta\in[0,2\pi)} \sin^9(\theta) = -1 \quad \rightsquigarrow \quad u + 1 \geqslant 0 \text{ in } D_R,$$

and the answer is yes, the problem admits a solution.

3. Consider the following Neumann problem (Laplace equation with fixed normal derivative on the boundary):

$$\begin{cases} \nabla^2 u = 0, & \text{in } D_R \\ \partial_n u(R,\theta) = \theta(2\pi - \theta)(\theta^2 - 12), & 0 \leqslant \theta \leqslant 2\pi \ (\text{ parametrising } \partial D_R) \end{cases}$$

a) Is there a solution?

Solution:
Let $A \subset \mathbb{R}^2$ be a (regular) region of the plane and the curve $\gamma = \partial A$ its boundary. As explained in the lecture notes, if u solves the Neumann problem on A

$$\begin{cases} \nabla^2 u = 0, & \text{in } A \\ \partial_n u = g, & \text{on } \gamma \end{cases}$$

then the integral of g on the boundary must vanish because of the divergence theorem

$$\int_\gamma g \, d\gamma = \int_\gamma (\partial_n u) \, d\gamma = \int_\gamma (\nabla u \cdot n) \, d\gamma = \int_A \operatorname{div}(\nabla u) \, dA = \int_A (\nabla^2 u) \, dA = \int_A 0 \, dA = 0.$$

In our case the region is a disk $A = D_R$ and the integral on the boundary is

$$\int_\gamma g \, d\gamma = \int_0^{2\pi} \theta(2\pi - \theta)(\theta^2 - 12) \, d\theta = \int_0^{2\pi} \left(-\theta^4 + 2\pi\theta^3 + 12\theta^2 - 24\pi\theta \right) d\theta =$$

$$= \left(-\frac{\theta^5}{5} + \frac{\pi\theta^4}{2} + 4\theta^3 - 12\pi\theta^2 \right) \Bigg|_0^{2\pi} = -\frac{32}{5}\pi^5 + 8\pi^5 + 32\pi^3 - 48\pi^3 =$$

$$= -16\pi^3 + \left(8 - \frac{32}{5} \right) \pi^5 = -16\pi^3 + \frac{8}{5}\pi^5 = \frac{8}{5}\pi^3(\pi^2 - 10) \neq 0.$$

This means that the problem is ill-posed and can't have a solution.

b) If the answer is yes, how many?

Solution: There is no solution.