Analysis III

Solutions Serie 2

1. Find the Laplace transform $F(s) := \mathscr{L}(f)(s)$ of the following functions

a) $f(t) = t^2 + 4t + 1$

Solution:
From the lecture we know that the function $g_n(t) = t^n$, $n \in \mathbb{N}$, has Laplace transform, defined for each $s > 0$: $G_n(s) = \frac{n!}{s^{n+1}}$. Using the linearity of $\mathscr{L}$ we have

$$F(s) = \mathscr{L}(t^2+4t+1)(s) = \mathscr{L}(t^2)(s) + 4\mathscr{L}(t)(s) + \mathscr{L}(1)(s) = \frac{2}{s^3} + 4 \cdot \frac{1}{s^2} + \frac{1}{s} = \\ = \frac{s^2+4s+2}{s^3}$$

b) $f(t) = \dfrac{1}{\sqrt{t}}$

Solution:
For each $s > 0$, we make the change of variable $u = st$, so that $dt = 1/s\, du$, and

$$F(s) = \int_0^{+\infty} t^{-1/2} e^{-st} dt = \frac{1}{s} \int_0^{+\infty} s^{1/2} u^{-1/2} e^{-u} du = \frac{1}{\sqrt{s}} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{s}}$$

c) $f(t) = \sin(\omega t), \quad \omega \in \mathbb{R}$

Solution 1:
The Laplace transform can be computed directly by definition, integrating by parts

$$\begin{aligned}
\mathscr{L}(\sin(\omega t))(s) &= \int_0^{+\infty} e^{-st} \sin(\omega t)\, dt = \cancel{-\frac{1}{s} e^{-st} \sin(\omega t)\Big|_0^{+\infty}} + \frac{\omega}{s} \int_0^{+\infty} e^{-st} \cos(\omega t)\, dt \\
&= -\frac{\omega}{s^2} e^{-st} \cos(\omega t)\Big|_0^{\infty} - \frac{\omega^2}{s^2} \int_0^{\infty} e^{-st} \sin(\omega t)\, dt \\
&= \frac{\omega}{s^2} - \frac{\omega^2}{s^2} \mathscr{L}(\sin(\omega t))(s) \\
&\Leftrightarrow \quad \mathscr{L}(\sin(\omega t))(s) = \frac{\omega}{s^2+\omega^2}
\end{aligned}$$

Solution 2 (slightly cheating - don't try this at home):
The Laplace transform has been defined just for real-valued functions. It can be defined in the same way for complex-valued functions, and also the variable s is assumed to be complex. Of course in the particular case of real-valued functions we have our old definition. From the lecture we know the Laplace transform of the real-valued exponential e^{at}, $a \in \mathbb{R}$, but also for the imaginary exponential $e^{i\omega t}$, $\omega \in \mathbb{R}$, the same computation makes sense and

$$\mathscr{L}\left(e^{i\omega t}\right)(s) = \frac{1}{s - i\omega} = \frac{s + i\omega}{s^2 + \omega^2} = \frac{s}{s^2 + \omega^2} + i\,\frac{\omega}{s^2 + \omega^2}, \qquad \mathfrak{Re}(s) > 0 \tag{1}$$

But $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$, thus by linearity

$$\mathscr{L}(e^{i\omega t}) = \mathscr{L}(\cos(\omega t)) + i\,\mathscr{L}(\sin(\omega t)). \tag{2}$$

Finally comparing real and imaginary parts of equations (1) and (2) we get

$$\mathscr{L}(\cos(\omega t))(s) = \frac{s}{s^2 + \omega^2}, \qquad \mathscr{L}(\sin(\omega t))(s) = \frac{\omega}{s^2 + \omega^2}. \tag{3}$$

d) $f(t) = \cos(\omega t), \quad \omega \in \mathbb{R}$

Solution 1:
We started computing the Laplace transform of $\sin(\omega t)$ with

$$\mathscr{L}(\sin(\omega t))(s) = \int_0^{+\infty} e^{-st}\sin(\omega t)\,dt = \frac{\omega}{s}\int_0^{+\infty} e^{-st}\cos(\omega t)\,dt = \frac{\omega}{s}\mathscr{L}(\cos(\omega t))(s)$$

from which we get

$$\mathscr{L}(\cos(\omega t))(s) = \frac{s}{s^2 + \omega^2}$$

Solution 2:
It's the strategy used above, from which we already got in equation (3) the expression for the Laplace transform of both sine and cosine.

e) $f(t) = \sin(\alpha t)\cos(\beta t), \quad \alpha, \beta \in \mathbb{R}$

Solution:
From Exercise **4.b)** of Serie 1 we know that

$$\sin(\alpha t)\cos(\beta t) = \frac{1}{2}\left(\sin((\alpha+\beta)t) + \sin((\alpha-\beta)t)\right)$$

and by Exercise **1.c)** of this Serie we know the Laplace transform of the sine. Thus

$$\mathscr{L}(\sin(\alpha t)\cos(\beta t))(s) = \frac{1}{2}\left(\frac{\alpha+\beta}{s^2 + (\alpha+\beta)^2} + \frac{\alpha-\beta}{s^2 + (\alpha-\beta)^2}\right)$$

2. Given a function $f(t)$ denote its Laplace transform by $F(s) := \mathscr{L}(f(t))(s)$. From the lecture we know that (for sufficiently nice functions) multiplication by t on the time-domain corresponds to derivative on the frequency-domain, that is

$$\mathscr{L}(tf(t))(s) = -\frac{d}{ds}F(s) \tag{1}$$

Using this property, prove that actually for each $n \in \mathbb{N}$:

$$\mathscr{L}(t^n f(t))(s) = (-1)^n \frac{d^n}{ds^n}F(s). \tag{2}$$

As an example, for $f(t) = 1$, we will find the known result

$$\mathscr{L}(t^n)(s) = \frac{n!}{s^{n+1}}.$$

Solution (by induction):
For $n = 0$ the equation is trivial and the case $n = 1$ is exactly equation (1). Suppose now $n \geqslant 2$ and by inductive hypothesis equation (2) holds for $n-1$ and for any function. Then we can apply it to $tf(t)$ and get

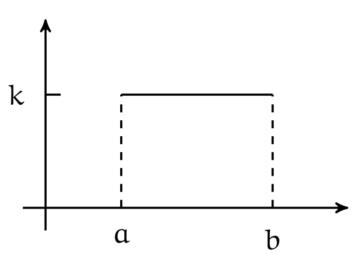
$$\mathscr{L}(t^n f(t))(s) = \mathscr{L}(t^{n-1} \cdot tf(t))(s) \overset{(2)}{=} (-1)^{n-1}\frac{d^{n-1}}{ds^{n-1}}\mathscr{L}(tf(t))(s) \overset{(1)}{=} (-1)^n \frac{d^n}{ds^n}F(s).$$

In particular for $f(t) = 1$ we have $\mathscr{L}(f)(s) = 1/s$ and then

$$\frac{d^n}{ds^n}\left(\frac{1}{s}\right) = (-1)^n \frac{n!}{s^{n+1}} \overset{(2)}{\Longrightarrow} \mathscr{L}(t^n)(s) = \frac{n!}{s^{n+1}}.$$

3. Find the Laplace transform of the following functions:

a)



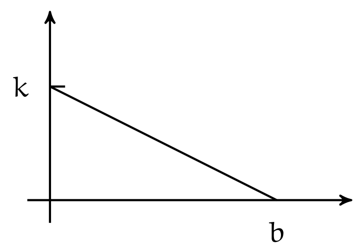
Solution:
We have

$$f(t) = \begin{cases} k, & a \leqslant t \leqslant b \\ 0, & \text{otherwise} \end{cases}$$

and then

$$\mathscr{L}(f)(s) = k\int_a^b e^{-st}\,dt = -\frac{k}{s}e^{-st}\Big|_a^b = \frac{k}{s}(e^{-sa} - e^{-sb})$$

b)



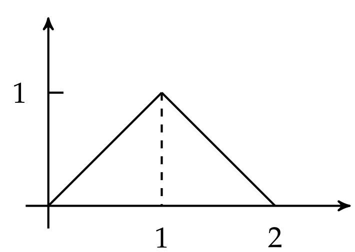
Solution:
We have

$$f(t) = \begin{cases} k(1-\frac{t}{b}), & 0 \leqslant t \leqslant b \\ 0, & \text{otherwise.} \end{cases}$$

Then, integrating by parts

$$\mathscr{L}(f)(s) = k\int_0^b e^{-st}\left(1-\frac{t}{b}\right)\,dt = -\frac{k}{s}e^{-st}\left(1-\frac{t}{b}\right)\Big|_0^b - \frac{k}{bs}\int_0^b e^{-st}\,dt$$
$$= \frac{k}{s} + \frac{k}{bs^2}e^{-st}\Big|_0^b = \frac{k}{s} + \frac{k}{bs^2}e^{-sb} - \frac{k}{bs^2} = \frac{k}{bs^2}\left(bs + e^{-sb} - 1\right)$$

c)



Solution:
We have

$$f(t) = \begin{cases} t, & 0 \leqslant t \leqslant 1 \\ 2-t, & 1 \leqslant t \leqslant 2 \\ 0, & \text{otherwise.} \end{cases}$$

Again, integrating by parts

$$\mathscr{L}(f)(s) = \int_0^1 e^{-st}t\,dt + \int_1^2 e^{-st}(2-t)\,dt = -\frac{t}{s}e^{-st}\Big|_0^1 + \frac{1}{s}\int_0^1 e^{-st}\,dt - \frac{2-t}{s}e^{-st}\Big|_1^2 - \frac{1}{s}\int_1^2 e^{-st}\,dt$$
$$= -\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-st}\Big|_0^1 + \frac{1}{s}e^{-s} + \frac{1}{s^2}e^{-st}\Big|_1^2 = -\frac{1}{s^2}e^{-s} + \frac{1}{s^2} + \frac{1}{s^2}e^{-2s} - \frac{1}{s^2}e^{-s}$$
$$= \frac{1}{s^2}\left(1 - 2e^{-s} + e^{-2s}\right) = \frac{(1-e^{-s})^2}{s^2}$$

Look at the next page!

4. Find the <u>inverse</u> Laplace transform $f = \mathscr{L}^{-1}(F)$ of

a) $F(s) = \dfrac{1}{s^4}$

Solution:

$$\mathscr{L}^{-1}\left(\frac{1}{s^4}\right) = \mathscr{L}^{-1}\left(\frac{1}{3!}\cdot\frac{3!}{s^4}\right) = \frac{1}{3!}\cdot\mathscr{L}^{-1}\left(\frac{3!}{s^4}\right) = \frac{1}{3!}t^3$$

b) $F(s) = \dfrac{1}{(s-8)^{10}}$

Solution:
Remember the s-shifting property, for which

$$\mathscr{L}(e^{at}f(t))(s) = \mathscr{L}(f)(s-a), \quad a \in \mathbb{R}.$$

But then

$$\mathscr{L}^{-1}\left(\frac{1}{(s-8)^{10}}\right) = \mathscr{L}^{-1}\left(\frac{1}{9!}\cdot\frac{9!}{(s-8)^{10}}\right) = \frac{1}{9!}\cdot\mathscr{L}^{-1}\left(\frac{9!}{(s-8)^{10}}\right) = \frac{1}{9!}e^{8t}t^9$$

c) $F(s) = \dfrac{s+3}{s^2-9}$

Solution:

$$\frac{s+3}{s^2-9} = \frac{1}{s-3} \implies \mathscr{L}^{-1}\left(\frac{s+3}{s^2-9}\right) = \mathscr{L}^{-1}\left(\frac{1}{s-3}\right) = e^{3t}$$

d) $F(s) = \dfrac{24}{(s-5)(s+3)}$

Solution:
By partial fraction decomposition

$$\frac{24}{(s-5)(s+3)} = \frac{24}{8}\left(\frac{1}{s-5} - \frac{1}{s+3}\right) = 3\left(\frac{1}{s-5} - \frac{1}{s+3}\right) \implies f(t) = 3(e^{5t} - e^{-3t})$$

e) $F(s) = \dfrac{1}{s^2+4}$

Solution:

$$\mathscr{L}^{-1}\left(\frac{1}{s^2+4}\right) = \mathscr{L}^{-1}\left(\frac{1}{2}\cdot\frac{2}{s^2+4}\right) = \frac{1}{2}\cdot\mathscr{L}^{-1}\left(\frac{2}{s^2+2^2}\right) = \frac{1}{2}\sin(2t)$$

f) **(*)** $F(s) = \dfrac{1}{s^2+4s+20}$

Please turn!

Solution:
We would like to write $s^2+4s+20$ in the form $(s+a)^2+\omega^2$. In fact this is possible for $a=2, \omega=4$, but then combining s-shifting property and the Laplace transform of the sine we get

$$\mathscr{L}^{-1}\left(\frac{1}{s^2+4s+20}\right)=\mathscr{L}^{-1}\left(\frac{1}{(s+2)^2+4^2}\right)=\frac{1}{4}e^{-2t}\sin(4t)$$

g) **(**)** $F(s)=\dfrac{s+1}{(s+2)(s^2+s+1)}$

Solution:
The strategy is firstly applying partial fraction decomposition and then trying to get some expression similar to the Laplace transforms of sine and cosine.

$$F(s)=\frac{s+1}{(s+2)(s^2+s+1)}=\frac{1}{3}\left(\frac{s+2}{s^2+s+1}-\frac{1}{s+2}\right)=\frac{1}{3}\left(\frac{s+2}{s^2+s+1}\right)-\frac{1}{3}\left(\frac{1}{s+2}\right)$$

The second term is the Laplace transform of $-1/3e^{-2t}$ so we just have to modify further the first term.

$$\frac{s+2}{s^2+s+1}=\frac{s+2}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}=\frac{s+2}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$=\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\sqrt{3}\cdot\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\implies$$

$$\implies\mathscr{L}^{-1}\left(\frac{s+2}{s^2+s+1}\right)=e^{-1/2t}\left(\cos\left(\frac{\sqrt{3}}{2}t\right)+\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

and finally the whole inverse Laplace transform will be

$$f=\mathscr{L}^{-1}\left(\frac{s+1}{(s+2)(s^2+s+1)}\right)=\frac{1}{3}e^{-1/2t}\left(\cos\left(\frac{\sqrt{3}}{2}t\right)+\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)-\frac{1}{3}e^{-2t}$$

h) **(**)** $F(s)=\dfrac{s}{(s-1)^2(s^2+2s+5)}$

Solution:
We still use first partial fraction decomposition to get terms which are known Laplace transforms (exponential, polynomials, sine and cosine).

$$\frac{s}{(s-1)^2(s^2+2s+5)}=\frac{1}{16}\cdot\frac{1}{s-1}+\frac{1}{8}\cdot\frac{1}{(s-1)^2}-\frac{1}{16}\cdot\frac{s}{s^2+2s+5}-\frac{5}{16}\cdot\frac{1}{s^2+2s+5}$$
$$=\frac{1}{16}\cdot\frac{1}{s-1}+\frac{1}{8}\cdot\frac{1}{(s-1)^2}-\frac{1}{16}\cdot\frac{s+1-1}{(s+1)^2+2^2}-\frac{5}{16}\cdot\frac{1}{(s+1)^2+2^2}=$$
$$=\frac{1}{16}\cdot\frac{1}{s-1}+\frac{1}{8}\cdot\frac{1}{(s-1)^2}-\frac{1}{16}\cdot\frac{s+1}{(s+1)^2+2^2}-\frac{4}{16}\cdot\frac{1}{(s+1)^2+2^2}\implies$$
$$\implies f=\mathscr{L}^{-1}\left(\frac{s}{(s-1)^2(s^2+2s+5)}\right)=\frac{e^t}{16}+\frac{te^t}{8}-\frac{1}{16}e^{-t}\left(\cos(2t)+2\sin(2t)\right)$$

5. (Bonus exercise)

a) (For those who have never seen this)
Exercise **1.b)** has been asked to solve using that $\Gamma(1/2) = \sqrt{\pi}$, and this exercise proves it. Let's call $I := \Gamma(1/2)$ this value.

(i) Use an opportune change of variables to prove that

$$I = 2\int_0^{+\infty} e^{-x^2} dx$$

Solution:

$$I = \Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-1/2} e^{-t} dt \overset{t=x^2}{=} \int_0^{+\infty} x^{-1} e^{-x^2} 2x dx = 2\int_0^{+\infty} e^{-x^2} dx$$

(ii) Note that

$$2\int_0^{+\infty} e^{-x^2} dx = \int_{-\infty}^{+\infty} e^{-x^2} dx$$

Solution:
The function we are integrating is symmetric with respect to $x = 0$. Thus the integral over all real numbers is 2 times the integral from 0 to $+\infty$.

(iii) Compute this integral in a smart way by computing its square. Fill the dots to get

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2} dy\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy =$$
$$= \cdots = \pi$$

Solution:
We fill the dots using polar coordinates, for which $dxdy = r dr d\varphi$:

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2} dy\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy =$$
$$= \int_0^{+\infty}\int_0^{2\pi} e^{-r^2} r dr d\varphi = 2\pi \int_0^{+\infty} e^{-r^2} r dr = 2\pi \cdot \left(-\frac{1}{2} e^{-r^2}\right)\Bigg|_0^{+\infty} = \pi$$

From (iii) must be either $I = \pm\sqrt{\pi}$. But I is obtained by integrating a positive function, therefore it must be the positive value $I = \sqrt{\pi}$.

b) Laplace transform of a finite linear combination of functions is the linear combination of their Laplace transforms:

$$\mathscr{L}(a_1 f_1 + \ldots a_m + f_m) = a_1 \mathscr{L}(f_1) + \ldots a_m \mathscr{L}(f_m), \quad a_1, \ldots, a_m \in \mathbb{R}$$

The same thing is true for infinite linear combination of functions under opportune conditions of convergence which are all satisfied in the following cases. To explain better, let's pretend we don't know the Laplace transform of the exponential and let's compute it explicitely from its power series expression

$$\mathscr{L}(e^{at})(s) = \mathscr{L}\left(\sum_{k=0}^{+\infty} \frac{(at)^k}{k!}\right)(s) = \sum_{k=0}^{+\infty} \frac{a^k}{k!}\mathscr{L}(t^k)(s) =$$
$$= \sum_{k=0}^{+\infty} \frac{a^k}{\cancel{k!}} \cdot \frac{\cancel{k!}}{s^{k+1}} = \frac{1}{s}\sum_{k=0}^{+\infty}\left(\frac{a}{s}\right)^k = \frac{1}{s}\cdot\frac{1}{1-\frac{a}{s}} = \frac{1}{s-a}$$

(i) Find again $\mathscr{L}(\sin(\omega t))(s)$ using the power series expansion

$$\sin(\omega t) = \sum_{k=0}^{+\infty}(-1)^k \frac{(\omega t)^{2k+1}}{(2k+1)!}$$

and verify that the result is the same already found in Exercise **1.c)**.

Solution:
We have

$$\mathscr{L}(\sin(\omega t)(s) = \mathscr{L}\left(\sum_{k=0}^{+\infty}(-1)^k \frac{(\omega t)^{2k+1}}{(2k+1)!}\right)(s) = \sum_{k=0}^{+\infty}(-1)^k \frac{\omega^{2k+1}}{(2k+1)!}\mathscr{L}(t^{2k+1})(s) =$$
$$= \sum_{k=0}^{+\infty}(-1)^k \frac{\omega^{2k+1}}{\cancel{(2k+1)!}} \cdot \frac{\cancel{(2k+1)!}}{s^{2k+2}} = \frac{\omega}{s^2}\sum_{k=0}^{+\infty}\left(-\frac{\omega^2}{s^2}\right)^k = \frac{\omega}{s^2}\cdot\frac{1}{1+\frac{\omega^2}{s^2}} = \frac{\omega}{s^2+\omega^2}$$

(ii) Using the same technique, prove that[1]

$$\mathscr{L}\left(\frac{\sin(t)}{t}\right) = \arctan\left(\frac{1}{s}\right)$$

Solution:

$$\mathscr{L}\left(\frac{\sin(t)}{t}\right)(s) = \mathscr{L}\left(\sum_{k=0}^{+\infty}(-1)^k \frac{t^{2k}}{(2k+1)!}\right)(s) = \sum_{k=0}^{+\infty}(-1)^k \frac{1}{(2k+1)!}\mathscr{L}(t^{2k})(s) =$$
$$= \sum_{k=0}^{+\infty}(-1)^k \frac{1}{(2k+1)!}\cdot\frac{(2k)!}{s^{2k+1}} = \sum_{k=0}^{+\infty}(-1)^k \frac{1}{(2k+1)}\cdot\frac{1}{s^{2k+1}} = \arctan\left(\frac{1}{s}\right)$$

[1]The power series expansion of the arctangent is

$$\arctan(x) = \sum_{k=0}^{+\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)}.$$