D-MAVT
D-MATL

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Analysis III

Solutions Serie 4

1. Find the inverse Laplace transform of the following functions.

a) $F(s) = \dfrac{e^{-2s}}{s^2+4}$

Solution:
The function
$$G(s) = \frac{1}{s^2+4}$$
is the Laplace transform of
$$g(t) = \frac{1}{2}\sin(2t)$$
and it's multiplied by some exponential, thus we can apply the t-shifting theorem and find
$$\frac{e^{-2s}}{s^2+4} = e^{-2s}G(s) = \mathscr{L}(u(t-2)g(t-2))(s)$$
from which
$$f(t) = \frac{1}{2}u(t-2)\sin(2(t-2)).$$

b) $F(s) = \dfrac{e^{-s}}{(s+1)^3}$

Solution:
As before we want to recognize the term multiplied by the exponential as the Laplace transform of some function. We have
$$G(s) = \frac{1}{s^3}, \quad g(t) = \frac{t^2}{2!}$$
and so by the s-shifting theorem
$$\frac{1}{(s+1)^3} = G(s+1) = \mathcal{L}(e^{-t}g(t))(s).$$
So, with $h(t) = e^{-t}g(t)$, using the t-shifting theorem
$$\frac{e^{-s}}{(s+1)^3} = e^{-s}G(s+1) = e^{-s}\mathscr{L}(e^{-t}g(t))(s) = e^{-s}\mathscr{L}(h(t))(s) = \mathscr{L}(u(t-1)h(t-1))(s)$$
and the final inverse transform is
$$f(t) = u(t-1)h(t-1) = u(t-1)\frac{e^{-(t-1)}(t-1)^2}{2}.$$

c) $F(s) = \dfrac{1}{s(s^2+1)}$

Solution 1:
We can use the integration property

$$\mathcal{L}\left(\int_0^t h(x)\,dx\right) = \frac{1}{s}\mathcal{L}\left(h(t)\right) \quad \Leftrightarrow \quad \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(h(t))\right) = \int_0^t h(x)\,dx$$

with $h(x) = \sin(x)$ and we get

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(\sin(t))\right) = \int_0^t \sin(x)\,dx = 1 - \cos(t)$$

Solution 2:
We can also use the property (2) of convolution because we have a product of two functions we can recognize as Laplace transform of some other functions. At the end we are obviously lead to the same integral

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right) = \mathcal{L}^{-1}\left(\mathcal{L}(1)\mathcal{L}(\sin(t))\right) = 1 \star \sin(t) = \int_0^t \sin(\tau)\,d\tau = 1 - \cos(t).$$

d) $F(s) = \dfrac{1}{(s^2+1)^2}$

Solution:
We have, with $h(t) = \sin(t)$,

$$F(s) = \frac{1}{(s^2+1)^2} = \mathcal{L}(h)\mathcal{L}(h)$$

and therefore

$$f(t) = \mathcal{L}^{-1}(F)(t) = (h \star h)\,(t)$$

which can be computed explicitely. For this is useful to remind from Exercise **4.b)** of Serie 1 that

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right) \tag{\#}$$

and

$$(h \star h)\,(t) = \int_0^t \sin(\tau)\sin(t-\tau)\,d\tau \overset{(\#)}{=} \frac{1}{2}\int_0^t \cos(2\tau - t) - \cos(t)\,d\tau =$$

$$= \frac{1}{2}\left.\frac{\sin(2\tau-t)}{2}\right|_0^t - \frac{1}{2}t\cos(t) = \frac{1}{2}\left(\frac{\sin(t)}{2} - \frac{\sin(-t)}{2}\right) - \frac{1}{2}t\cos(t) =$$

$$= \frac{1}{2}\sin(t) - \frac{1}{2}t\cos(t) = \frac{1}{2}\left(\sin(t) - t\cos(t)\right)$$

Look at the next page!

2. Compute the following convolutions.

a) $e^{at} \star e^{bt} \quad (a, b \in \mathbb{R})$

Solution 1 (direct computation):
Let's first consider the case $a \neq b$, in which

$$e^{at} * e^{bt} = \int_0^t e^{a\tau} e^{b(t-\tau)}\, d\tau = e^{bt} \int_0^t e^{(a-b)\tau}\, d\tau = \frac{e^{bt}}{a-b} e^{(a-b)\tau}\Big|_0^t$$
$$= \frac{e^{bt}}{a-b}(e^{(a-b)t} - 1) = \frac{e^{at} - e^{bt}}{a-b}.$$

Instead for $a = b$ we have

$$e^{at} \star e^{at} = \int_0^t e^{a\tau} e^{a(t-\tau)} d\tau = e^{at} \int_0^t d\tau = t\, e^{at}.$$

Solution 2 (Laplace transform):
We can use property (2) of Laplace transform to get

$$e^{at} \star e^{bt} = \mathcal{L}^{-1}\left(\mathcal{L}(e^{at})\mathcal{L}(e^{bt})\right) = \mathcal{L}^{-1}\left(\frac{1}{s-a} \cdot \frac{1}{s-b}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s-a)(s-b)}\right)$$

Now in the case $a \neq b$ we use partial fraction decomposition and

$$\frac{1}{(s-a)(s-b)} = \frac{1}{a-b}\left(\frac{1}{s-a} - \frac{1}{s-b}\right) \quad \Longrightarrow \quad e^{at} \star e^{bt} = \frac{e^{at} - e^{bt}}{a-b}$$

while in the case $a = b$

$$\frac{1}{(s-a)^2} = \mathcal{L}\left(te^{at}\right) \quad \Longrightarrow \quad e^{at} \star e^{at} = t\, e^{at}.$$

b) $\sin(t) \star \cos(t)$

Solution:
With the trigonometric identity found in Exercise **4.b)** of Serie 1

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right) \qquad (\star)$$

we have

$$\sin(t) \star \cos(t) = \int_0^t \sin(\tau)\cos(t-\tau)\, d\tau \overset{(\star)}{=} \frac{1}{2}\int_0^t \sin(t)\, d\tau + \frac{1}{2}\int_0^t \sin(2\tau - t)\, d\tau =$$
$$= \frac{1}{2} t\sin(t) - \frac{\cos(2\tau - t)}{4}\Big|_{\tau=0}^{\tau=t} = \frac{1}{2}t\sin(t) - \frac{1}{4}\left(\cancel{\cos(t) - \cos(-t)}\right) = \frac{1}{2}t\sin(t).$$

 Please turn!

3. Solve the following IVP:

$$\begin{cases} y'' + 5y' + 6y = \delta(t - \frac{\pi}{2}) + u(t-\pi)\cos(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

Solution:
The Laplace transform of the left-hand side is

$$s^2Y(s) - sy(0) - y'(0) + 5sY(s) - 5y(0) + 6Y(s) = (s^2+5s+6)Y(s) = (s+2)(s+3)Y(s),$$

and for the Laplace transform of the right-hand side we remember that

$$\mathscr{L}(\delta(t-a))(s) = e^{-as}$$

and

$$\cos(t) = -\cos(t-\pi),$$

so, using also the t-shifting we get

$$\mathscr{L}(\delta(t-\tfrac{\pi}{2}) + u(t-\pi)\cos(t))(s) = e^{-\pi s/2} - e^{-\pi s}\left(\frac{s}{s^2+1}\right).$$

Therefore, solving for $Y(s)$,

$$\begin{aligned} Y(s) &= e^{-\pi s/2}\left(\frac{1}{(s+3)(s+2)}\right) - e^{-\pi s}\left(\frac{s}{(s+3)(s+2)(s^2+1)}\right) \\ &= e^{-\pi s/2}\left(\frac{1}{(s+2)} - \frac{1}{(s+3)}\right) - e^{-\pi s}\left(\frac{3}{10(s+3)} - \frac{2}{5(s+2)} + \frac{s+1}{10(s^2+1)}\right) \\ &= e^{-\pi s/2}\mathscr{L}(e^{-2t} - e^{-3t})(s) - \frac{e^{-\pi s}}{10}\mathscr{L}(3e^{-3t} - 4e^{-2t} + \cos(t) + \sin(t))(s) \end{aligned}$$

and finally from the t-shifting theorem follows

$$\begin{aligned} y(t) &= u(t-\tfrac{\pi}{2})(e^{-2t+\pi} - e^{-3t+3\pi/2}) - \frac{1}{10}u(t-\pi)(3e^{-3t+3\pi} - 4e^{-2t+2\pi} + \cos(t-\pi) + \sin(t-\pi)) \\ &= u(t-\tfrac{\pi}{2})(e^{-2t+\pi} - e^{-3t+3\pi/2}) - \frac{1}{10}u(t-\pi)(3e^{-3t+3\pi} - 4e^{-2t+2\pi} - \cos(t) - \sin(t)). \end{aligned}$$

Look at the next page!

4. (Bonus exercise)

a) Use the Laplace transform to verify that the solution of the following IVP

$$\begin{cases} y' = g(t) \\ y(0) = c \end{cases}$$

is the obvious

$$y(t) = c + \int_0^t g(\tau)\,d\tau$$

Solution:

Computing the Laplace transform of both sides of the differential equation we get

$$sY(s) - y(0) = G(s) \quad \Leftrightarrow \quad Y(s) = \frac{c}{s} + \frac{G(s)}{s}$$

To transform back the second summand of the right-hand side we remember again the integration property

$$\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right) = \int_0^t g(\tau)\,d\tau$$

from which we get

$$y(t) = c + \int_0^t g(\tau)\,d\tau$$

b) Use the Laplace transform to verify that the solution of the following IVP

$$\begin{cases} y'' + y = g(t) \\ y(0) = c \\ y'(0) = d \end{cases}$$

is

$$y(t) = c\cos(t) + d\sin(t) + \int_0^t \sin(\tau)g(t-\tau)\,d\tau$$

Solution:

$$\begin{aligned} & s^2Y(s) - sy(0) - y'(0) + Y(s) = G(s) \qquad \Leftrightarrow \\ \Leftrightarrow \qquad & Y(s) = \frac{cs+d}{s^2+1} + \frac{G(s)}{s^2+1} = c\frac{s}{s^2+1} + d\frac{1}{s^2+1} + \frac{G(s)}{s^2+1} \end{aligned}$$

The first summand of the right-hand is an opportune combination of Laplace transforms of cosine and sine. Instead, to transform back the second summand of the right-hand side we need to use property (2) of convolution and get

$$\mathcal{L}^{-1}\left(\frac{G(s)}{s^2+1}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) \star g(t) = \sin(t) \star g(t) = \int_0^t \sin(\tau)g(t-\tau)\,d\tau$$

from which

$$y(t) = c\cos(t) + d\sin(t) + \int\limits_0^t \sin(\tau)g(t-\tau)\,d\tau$$