D-MAVT
D-MATL

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Analysis III

Solutions Serie 6

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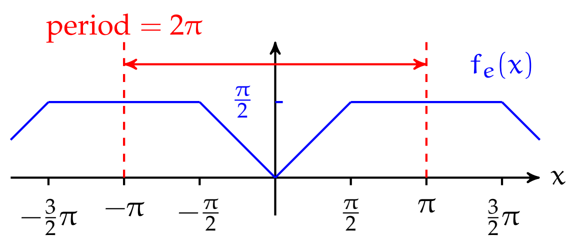
1. Consider the function

$$f(x) = \begin{cases} x, & 0 \leqslant x \leqslant \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} \leqslant x \leqslant \pi \end{cases}$$

a) Extend f to an even function on the interval $[-\pi, \pi]$ and then finally to an even, 2π-periodic function on $\mathbb{R}$ and call this function f_e.
Sketch the graph of f_e and find its Fourier series.

Solution:
The even extension f_e is given, in the interval $[-\pi, \pi]$, by

$$f_e(x) = \begin{cases} \frac{\pi}{2}, & -\pi \leqslant x \leqslant -\frac{\pi}{2} \\ -x, & -\frac{\pi}{2} \leqslant x \leqslant 0 \\ x, & 0 \leqslant x \leqslant \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} \leqslant x \leqslant \pi. \end{cases}$$



Being even, the b_n coefficients will vanish, while

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_e(x)\,dx = \frac{1}{\pi}\int_0^{\pi} f_e(x)\,dx = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} x\,dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}\,dx = \frac{1}{2\pi}x^2\Big|_0^{\frac{\pi}{2}} + \frac{x}{2}\Big|_{\frac{\pi}{2}}^{\pi} = \frac{3\pi}{8},$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_e(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} f_e(x)\cos(nx)\,dx = \\
&= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\cos(nx)\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}\cos(nx)\,dx = \\
&= \frac{2}{\pi}\left(\frac{x}{n}\sin(nx)\Big|_0^{\frac{\pi}{2}} - \frac{1}{n}\int_0^{\frac{\pi}{2}}\sin(nx)\,dx\right) + \frac{1}{n}\sin(nx)\Big|_{\frac{\pi}{2}}^{\pi} = \\
&= \frac{1}{n}\sin\left(\tfrac{n\pi}{2}\right) + \frac{2}{n^2\pi}\cos(nx)\Big|_0^{\frac{\pi}{2}} + \frac{1}{n}\sin(nx)\Big|_{\frac{\pi}{2}}^{\pi} = \\
&= \frac{2}{n^2\pi}\left(\cos\left(\tfrac{n\pi}{2}\right) - 1\right) = \begin{cases} -\frac{2}{n^2\pi}, & n = 2j+1 \\ \frac{2}{n^2\pi}((-1)^j - 1), & n = 2j. \end{cases}
\end{aligned}
$$

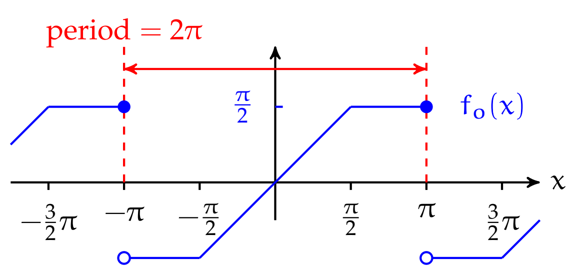
The Fourier series is thus

$$
\frac{3\pi}{8} + \frac{2}{\pi}\sum_{j=1}^{+\infty}\frac{1}{(2j)^2}((-1)^j - 1)\cos(2jx) - \frac{2}{\pi}\sum_{j=0}^{+\infty}\frac{1}{(2j+1)^2}\cos((2j+1)x).
$$

b) Do the same for the odd, 2π-periodic extension[1] of f (call this f_o).

Solution:
The odd extension f_o is given, in the interval $(-\pi, \pi]$, by

$$
f_o(x) = \begin{cases} -\frac{\pi}{2}, & -\pi < x \leqslant -\frac{\pi}{2} \\ x, & -\frac{\pi}{2} \leqslant x \leqslant 0 \\ x, & 0 \leqslant x \leqslant \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} \leqslant x \leqslant \pi. \end{cases}
$$



[1]to be precise, we can't extend f to an odd, periodic function everywhere. In fact by periodicity and oddness we should have $f_o(-\pi) = f_o(\pi) = -f_o(-\pi)$, and therefore $f_o(\pm\pi) = 0$, while $f(\pi) = \pi$. The points in which there is a doubt about what value to assign to this new function are the odd integer multiples of π. Let's assign to these points the value π just to fix the convention, at the end - as you can observe - nothing will be depend on the choice of this value, and we could have also let f_o not defined.

Therefore here the a_n coefficients will be all zero, while

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_o(x)\sin(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f_o(x)\sin(nx)\,dx = \\
&= \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} x\sin(nx)\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}\sin(nx)\,dx \\
&= \frac{2}{\pi}\left(-\frac{x}{n}\cos(nx)\Big|_0^{\frac{\pi}{2}} + \frac{1}{n}\int_0^{\frac{\pi}{2}}\cos(nx)\,dx\right) - \frac{1}{n}\cos(nx)\Big|_{\frac{\pi}{2}}^{\pi} \\
&= -\frac{1}{n}\cos\left(\tfrac{n\pi}{2}\right) + \frac{2}{n^2\pi}\sin(nx)\Big|_0^{\frac{\pi}{2}} - \frac{1}{n}\cos(n\pi) + \frac{1}{n}\cos\left(\tfrac{n\pi}{2}\right) \\
&= \frac{2}{n^2\pi}\sin\left(\tfrac{n\pi}{2}\right) - \frac{1}{n}\cos(n\pi) = \begin{cases} -\frac{1}{n}, & n = 2j \\ \frac{2}{n^2\pi}(-1)^j + \frac{1}{n}, & n = 2j+1 \end{cases}
\end{aligned}$$

and the Fourier series is

$$-\sum_{j=1}^{+\infty}\frac{1}{2j}\sin(2jx) + \sum_{j=0}^{+\infty}\left(\frac{2}{(2j+1)^2\pi}(-1)^j + \frac{1}{2j+1}\right)\sin((2j+1)x).$$

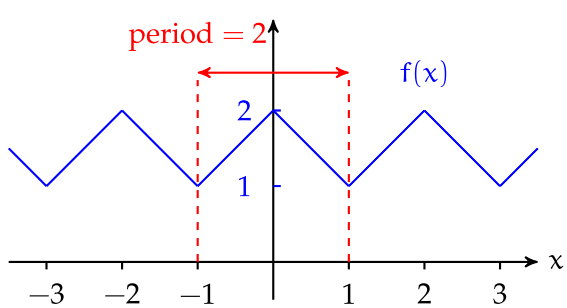
2. Consider the function x in the interval $[1,2]$ and extend it to an even, 2-periodic function f on $\mathbb{R}$.

a) Sketch the graph of f and find its Fourier series.

Solution:
Analogously to what happened in the previous exercise, we are interested only in the formula for f in the interval $[0,1]$ to find its Fourier coefficients. We have

$$x \in [0,1] \quad \Longrightarrow \quad f(x) = f(-x) = f(-x+2) = -x+2.$$

The graph is the following



The a_n coefficients will be

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \int_0^1 (2-x)\,dx = 2x - \frac{x^2}{2}\Big|_0^1 = \frac{3}{2}$$

and

$$\begin{aligned} a_n &= \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = 2\int_0^1 (2-x)\cos(n\pi x)\,dx \\ &= \frac{4}{n\pi}\sin(n\pi x)\Big|_0^1 - 2\left(\frac{x}{n\pi}\sin(n\pi x)\Big|_0^1 - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)\,dx\right) \\ &= -\frac{2}{n^2\pi^2}\cos(n\pi x)\Big|_0^1 = \frac{2}{n^2\pi^2}(1-(-1)^n) = \begin{cases} 0, & n = 2j \\ \frac{4}{n^2\pi^2}, & n = 2j+1 \end{cases}. \end{aligned}$$

So the Fourier series will be

$$\frac{3}{2} + \frac{4}{\pi^2}\sum_{j\geqslant 0} \frac{1}{(2j+1)^2}\cos((2j+1)\pi x).$$

b) Will the Fourier series converge pointwise to the function f everywhere? (Justify your answer using what learnt in the script).

Solution:
The extended function f is everywhere continuous and has left and right derivatives everywhere[2]. Therefore the Fourier series will converge to it in every point, and it's legitimate to write

$$f(x) = \frac{3}{2} + \frac{4}{\pi^2}\sum_{j\geqslant 0} \frac{1}{(2j+1)^2}\cos((2j+1)\pi x).$$

because indeed, the equality between these two functions hold for every $x \in \mathbb{R}$.

3. Let f be the 2L-periodic extension of x from $[-L, L)$ to the whole real line as in Exercise **3.** of Serie 5. Find its complex Fourier series

$$\sum_{n=-\infty}^{+\infty} c_n e^{i\frac{n\pi}{L}x}$$

Verify that the coefficients c_n of this serie are related to the real coefficients a_n, b_n as in the script.
<u>If you have not computed it before:</u> the real Fourier series of f is

$$\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{2L}{\pi n}\sin\left(\frac{n\pi}{L}x\right) \quad \rightsquigarrow \quad \begin{cases} a_n = 0 \\ b_n = (-1)^{n+1}\frac{2L}{\pi n} \end{cases}$$

[2] although these left and right derivatives don't agree in the integer points $x = k \in \mathbb{Z}$, but that's not relevant

Solution:
The complex Fourier coefficients for f are, for $n \neq 0$,

$$
\begin{aligned}
c_n &= \frac{1}{2L}\int_{-L}^{L} x e^{-i\frac{n\pi}{L}x}\,dx = \frac{L}{2\pi^2}\int_{-\pi}^{\pi} y e^{-iny}\,dy = \\
&= \frac{L}{2\pi^2}\left(-\frac{y}{in}e^{-iny}\Big|_{-\pi}^{\pi} + \frac{1}{in}\int_{-\pi}^{\pi} e^{-iny}\,dy\right) = \\
&= \frac{L}{2\pi^2}\left(-\frac{\pi}{in}e^{-in\pi} - \frac{\pi}{in}e^{in\pi} + \frac{1}{n^2}e^{-iny}\Big|_{-\pi}^{\pi}\right) = \\
&= \frac{L}{2\pi^2}\left(-\frac{\pi}{in}e^{-in\pi} - \frac{\pi}{in}e^{in\pi} + \frac{1}{n^2}e^{-in\pi} - \frac{1}{n^2}e^{in\pi}\right) = \\
&= \frac{(-1)^n L}{2\pi^2}\left(-\frac{\pi}{in} - \frac{\pi}{in} + \frac{1}{n^2} - \frac{1}{n^2}\right) \\
&= -\frac{(-1)^n L}{in\pi} = i\,\frac{(-1)^n L}{n\pi}
\end{aligned}
$$

and for $n = 0$ is

$$c_0 = \frac{1}{2L}\int_{-L}^{L} x\,dx = \frac{x^2}{4L}\Big|_{-L}^{L} = 0.$$

Therefore the complex Fourier series of f is

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} i\,\frac{(-1)^n L}{n\pi} e^{inx}.$$

The formula relating the real coefficients to the complex coefficients is

$$
\begin{cases}
a_0 = c_0 \\
a_n = c_n + c_{-n} \quad (n \geqslant 1) \\
b_n = i\,(c_n - c_{-n})
\end{cases}
$$

and substituting we get indeed

$$
\begin{cases}
a_0 = c_0 = 0 \\
a_n = c_n + c_{-n} = i\,\frac{(-1)^n L}{n\pi} - i\,\frac{(-1)^n L}{n\pi} = 0 \\
b_n = i(c_n - c_{-n}) = i\left(i\,\frac{(-1)^n L}{n\pi} + i\,\frac{(-1)^n L}{n\pi}\right) = (-1)^{n+1}\frac{2L}{n\pi}
\end{cases}
$$

which is what we expected.

4. Consider again the 2L-periodic extension of x as in the previous Exercise. Find the minimum value E_N^* of the square error at the step N, which is

$$E_N^* = \int_{-L}^{L} f^2\,dx - L\left(2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right), \qquad a_0, a_n, b_n \ \text{ Fourier coefficients.}$$

(To check that your computation is correct) prove that

$$\lim_{N\to+\infty} E_N^* = \frac{2}{3}L^3 - 4\frac{L^3}{\pi^2}\sum_{n=1}^{+\infty}\frac{1}{n^2}$$

Solution:
The coefficients have been recalled in the previous exercise and are

$$\begin{cases} a_n = 0 \\ b_n = (-1)^{n+1}\frac{2L}{\pi n} \end{cases}$$

Therefore

$$E_N^* = \int_{-L}^{L} x^2\,dx - L\sum_{n=1}^{N} b_n^2 = \frac{2}{3}L^3 - L\sum_{n=1}^{N}\frac{4L^n}{\pi^2 n^2}$$

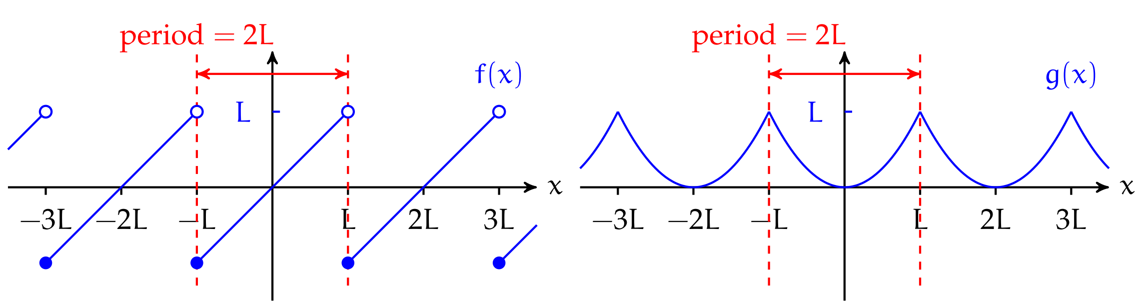
which in fact has limit for $N \to +\infty$ equal to

$$\frac{2}{3}L^3 - 4\frac{L^3}{\pi^2}\sum_{n=1}^{+\infty}\frac{1}{n^2}$$

5. Let f be any 2L-periodic function. From the script we know that if f is well behaved (for example everywhere continuous except a discrete set of points and with left and right derivatives at every point) then, calling by F its Fourier series, we have for every point x_0

$$F(x_0) = \frac{1}{2}\left(f^+(x_0) + f^-(x_0)\right), \qquad \text{where} \;\; f^\pm(x_0) = \lim_{x\to x_0^\pm} f(x) = \lim_{\epsilon\to 0^+} f(x_0 \pm \epsilon)$$

In particular if f is continuous in x_0 then $F(x_0) = f(x_0)$ because left and right limit of f coincide.[3]
Let now f and g be, respectively, the 2L-periodic extensions to $\mathbb{R}$ of x and x^2 from $[-L, L)$. Sketch a graph of these functions.



[3]This gives an answer to Exercise **2.b)**.

Look at the next page!

a) Are f and g well behaved in the sense specified above?

Solution:
Yes. f is continuous everywhere except in the odd integer multiples of L (so, anyway, a discrete set of points), and has left and right derivatives everywhere. g is continuous everywhere and has left and right derivatives everywhere.

b) What are the points of discontinuity of f and g?

Solution:
As said before, g is continuous everywhere while f has discontinuities in the points $x_k = kL$, with k odd integer, that is $x = L, -L, 3L, -3L, \ldots$.

c) What are the mean values of the left and right limit of f in its points of discontinuity?

$$\frac{1}{2}\left(f^+(x_0) + f^-(x_0)\right) = \ ?$$

Solution:
In each of these points the right limit is always $-L$ while the left limit is L, therefore the mean value is

$$\frac{1}{2}\left(f^+(x_k) + f^-(x_k)\right) = \frac{1}{2}(-L + L) = 0$$

d) Does the Fourier series F of f converge to these values in these points? If the answer to **5.a)** is affirmative, then yes.
Verify it explicitely.

Solution:
The answer to **5.a)** is indeed affirmative so we should have that the Fourier series

$$F(x) = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{2L}{\pi n} \sin\left(\frac{n\pi}{L} x\right)$$

converges to zero in the points $x_k = kL$ with k odd integer. Indeed for these points

$$\sin\left(\frac{n\pi}{L} x_k\right) = \sin\left(\frac{n\pi}{L} kL\right) = \sin(n\pi k) = 0 \qquad \Longrightarrow \qquad F(x_k) = 0.$$

e) Prove that the Fourier series of g is

$$G(x) = \frac{L^2}{3} + \sum_{n=1}^{+\infty} (-1)^n \frac{4L^2}{\pi^2 n^2} \cos\left(\frac{n\pi}{L} x\right)$$

Solution:
g is an even function, so it will have just a_n coefficients. We have

$$a_0 = \frac{1}{2L} \int_{-L}^{L} g(x)\, dx = \frac{1}{L} \int_{0}^{L} x^2\, dx = \frac{L^2}{3}$$

The other a_n coefficients can be either found explicitely by integrating twice by parts (left to you), or using the following trick.
The b_n coefficients of the derivative g' are related to our coefficients by =

$$
\begin{aligned}
b_n(g') &= \frac{1}{L}\int_{-L}^{L} g'(x)\sin\left(\frac{n\pi}{L}x\right) dx = \\
&= \frac{1}{L}\left(\cancel{g(x)\sin\left(\frac{n\pi}{L}x\right)\Big|_{-L}^{L}} - \frac{n\pi}{L}\int_{-L}^{L} g(x)\cos\left(\frac{n\pi}{L}x\right) dx\right) = \\
&= -\frac{n\pi}{L}a_n(g)
\end{aligned}
$$

Reading this equality from right to left, we can use that $b_n(g') = 2\,b_n(f)$ and obtain

$$a_n(g) = -\frac{L}{n\pi}b_n(g') = -\frac{2L}{n\pi}b_n(f) = (-1)^n\frac{4L^2}{\pi^2 n^2}$$

from which

$$G(x) = \frac{L^2}{3} + \sum_{n=1}^{+\infty}(-1)^n\frac{4L^2}{\pi^2 n^2}\cos\left(\frac{n\pi}{L}x\right)$$

6. **(Bonus exercise)** With the same notations of the previous exercise.

   a) Because g (the 2L-periodic extension of x^2) is well-behaved and continuous everywhere, its Fourier series G converge to it in every point. In particular

   $$L^2 = g(L) = G(L).$$

   Deduce from this equality the value of the Riemann Zeta function in 2

   $$\zeta(2) := \sum_{n=1}^{+\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

   Solution:
   From the previous exercise

   $$
   \begin{aligned}
   L^2 = G(L) &= \frac{L^2}{3} + \sum_{n=1}^{+\infty}(-1)^n\frac{4L^2}{\pi^2 n^2}\cos\left(\frac{n\pi}{\cancel{L}}\cancel{L}\right) = \frac{L^2}{3} + \sum_{n=1}^{+\infty}\frac{4L^2}{\pi^2 n^2} = \\
   &= \frac{L^2}{3} + \frac{4L^2}{\pi^2}\zeta(2) \quad\Longrightarrow\quad \zeta(2) = \frac{\pi^2}{4L^2}\left(L^2 - \frac{L^2}{3}\right) = \frac{\pi^2}{6}.
   \end{aligned}
   $$

   b) Use this value to deduce that the limit of the square error for f computed in Exercise **4.** is zero.

   Solution:

Now that we have given a name to the sum of all squares, we can read the limit of the square error found in Exercise **4.** as

$$\lim_{N\to+\infty} E_N^*(f) = \frac{2}{3}L^3 - 4\frac{L^3}{\pi^2}\sum_{n=1}^{+\infty}\frac{1}{n^2} = \frac{2}{3}L^3 - 4\frac{L^3}{\pi^2}\zeta(2) = \frac{2}{3}L^3 - \frac{2}{3}L^3 = 0.$$

c) Compute the square error for g, $E_N^*(g)$, and observe that the following are equivalent[4]

(i) $\lim_{N\to+\infty} E_N^*(g) = 0$

(ii) $\zeta(4) := \sum_{n=1}^{+\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$

Solution:
The coefficients for g are

$$\begin{cases} a_0 = \frac{L^2}{3} \\ a_n = (-1)^n\frac{4L^2}{\pi^2 n^2} \end{cases}$$

while the integral of its square is

$$\int_{-L}^{L} x^4 \, dx = \frac{2}{5}L^5$$

therefore

$$E_N^*(g) = \frac{2}{5}L^5 - L\left(\frac{2L^4}{9} + \sum_{n=1}^{N}\frac{16L^4}{\pi^4 n^4}\right) = \frac{8}{45}L^5 - \frac{16L^5}{\pi^4}\sum_{n=1}^{N}\frac{1}{n^4}$$

and

$$\lim_{N\to+\infty} E_N^*(g) = \frac{8}{45}L^5 - \frac{16L^5}{\pi^4}\sum_{n=1}^{+\infty}\frac{1}{n^4} = \frac{8}{45}L^5 - \frac{16L^5}{\pi^4}\zeta(4)$$

so, indeed,

$$\text{(i)} \quad \lim_{N\to+\infty} E_N^*(g) = 0 \quad \Leftrightarrow \quad \frac{8}{45}L^5 - \frac{16L^5}{\pi^4}\zeta(4) = 0 \quad \Leftrightarrow \quad \zeta(4) = \frac{\pi^4}{90} \quad \text{(ii)}.$$

Remark: In a similar way one can consider the periodic extension of x^k for any positive integer k, use the fact that the square error has limit zero (which is always true for nice functions like these), and deduce the values of the Zeta function on the even integers

$$\zeta(2k) = \frac{(-1)^{k+1}B_{2k}(2\pi)^{2k}}{2\cdot(2k)!}, \qquad B_{2k} = 2k\text{-th Bernoulli number.}$$

[4] in fact, they are (both) true.