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Analysis III

Solutions Serie 9

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1. For $0 < k < 1$, find (via Fourier series) the solution $u = u(x,t)$ of the 1-dimensional wave equation with the following boundary and initial conditions:

$$\begin{cases} u_{tt} = u_{xx}, & \\ u(0,t) = 0 = u(1,t), & t \geqslant 0 \\ u(x,0) = kx(1-x^2), & 0 \leqslant x \leqslant 1 \\ u_t(x,0) = 0, & 0 \leqslant x \leqslant 1 \end{cases}$$

Solution:
In this case $c = 1$, $L = 1$, $\lambda_n = c\frac{n\pi}{L} = n\pi$. The solution of the 1-dimensional wave equation via Fourier series is thus

$$u(x,t) = \sum_{n=1}^{+\infty} (B_n \cos(n\pi t) + B_n^* \sin(n\pi t)) \sin(n\pi x).$$

The coefficients are found by imposing the initial conditions. Firstly the initial position must be

$$u(x,0) = \sum_{n=1}^{+\infty} B_n \sin(n\pi x) = kx(1-x^2).$$

For this equality to hold the coefficients B_n must be the coefficients of the Fourier series of the odd, $2L$($= 2$ in this case)-periodic extension of the function $kx(1-x^2)$ from the interval $[0,1]$. That is:

$$B_n = 2\int_0^1 kx(1-x^2)\sin(n\pi x)\,dx = 2k\int_0^1 (x-x^3)\sin(n\pi x)\,dx =$$

$$= 2k\left(-(x-x^3)\frac{\cos(n\pi x)}{n\pi}\bigg|_0^1 + \frac{1}{n\pi}\int_0^1 (1-3x^2)\cos(n\pi x)\,dx \right) =$$

$$= \frac{2k}{n\pi}\left((1-3x^2)\frac{\sin(n\pi x)}{n\pi}\bigg|_0^1 + \frac{1}{n\pi}\int_0^1 6x\sin(n\pi x)\,dx \right) =$$

$$= \frac{12k}{n^2\pi^2}\left(-x\frac{\cos(n\pi x)}{n\pi}\bigg|_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx\right) =$$

$$= -\frac{12k}{n^3\pi^3}(-1)^n - \frac{12k}{n^4\pi^4}\sin(n\pi x)\bigg|_0^1 =$$

$$= \frac{12k}{n^3\pi^3}(-1)^{n+1}.$$

The initial speed instead gives trivially

$$u_t(x,0) = \sum_{n=1}^{+\infty} n\pi B_n^* \sin(n\pi x) = 0 \quad \Leftrightarrow \quad B_n^* = 0.$$

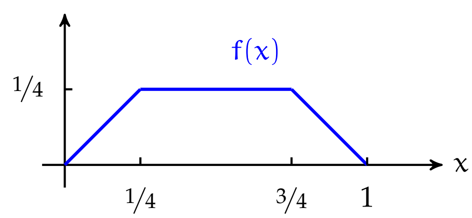
Finally, the solution is

$$u(x,t) = \frac{12k}{\pi^3}\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^3}\cos(n\pi t)\sin(n\pi x).$$

2. Find (via Fourier series) the solution $u = u(x,t)$ of the 1-dimensional wave equation with the following boundary and initial conditions:

$$\begin{cases} u_{tt} = u_{xx}, & \\ u(0,t) = 0 = u(1,t), & t \geqslant 0 \\ u(x,0) = f(x), & 0 \leqslant x \leqslant 1 \\ u_t(x,0) = 0, & 0 \leqslant x \leqslant 1 \end{cases}$$

where $f(x)$ is the following function



Solution:
The function described is

$$f(x) = \begin{cases} x, & 0 \leqslant x \leqslant 1/4 \\ 1/4, & 1/4 \leqslant x \leqslant 3/4 \\ 1-x, & 3/4 \leqslant x \leqslant 1 \end{cases}$$

Here we have again $c = 1$, $L = 1$, $\lambda_n = c\frac{n\pi}{L} = n\pi$, and the Fourier series solution will be

$$u(x,t) = \sum_{n=1}^{+\infty} \left(B_n\cos(n\pi t) + B_n^*\sin(n\pi t)\right)\sin(n\pi x)$$

 Look at the next page!

with, as before, $B_n^* = 0$ because the initial speed is zero.
Imposing the initial position

$$u(x,0) = \sum_{n=1}^{+\infty} B_n \sin(n\pi x) = f(x)$$

amounts to finding the coefficients of the odd, 2-periodic extension of f, that is

$$\begin{aligned}
B_n &= 2\int_0^1 f(x)\sin(n\pi x)\,dx = \\
&= 2\int_0^{1/4} x\sin(n\pi x)\,dx + \frac{1}{2}\int_{1/4}^{3/4} \sin(n\pi x)\,dx + 2\int_{3/4}^{1} (1-x)\sin(n\pi x)\,dx = \\
&= 2\left(-\frac{x\cos(n\pi x)}{n\pi}\bigg|_0^{1/4} + \frac{1}{n\pi}\int_0^{1/4}\cos(n\pi x)\,dx \right) - \frac{\cos(n\pi x)}{2n\pi}\bigg|_{1/4}^{3/4} + \\
&\quad + 2\left(-(1-x)\frac{\cos(n\pi x)}{n\pi}\bigg|_{3/4}^{1} - \frac{1}{n\pi}\int_{3/4}^{1}\cos(n\pi x)\,dx \right) = \\
&= -\frac{\cos\left(\frac{n\pi}{4}\right)}{2n\pi} + \frac{2\sin(n\pi x)}{n^2\pi^2}\bigg|_0^{1/4} - \frac{\cos\left(\frac{3n\pi}{4}\right)}{2n\pi} + \frac{\cos\left(\frac{n\pi}{4}\right)}{2n\pi} + \frac{\cos\left(\frac{3n\pi}{4}\right)}{2n\pi} - \frac{2\sin(n\pi x)}{n^2\pi^2}\bigg|_{3/4}^{1} \\
&= \frac{2}{n^2\pi^2}\left(\sin\left(\frac{n\pi}{4}\right) + \sin\left(\frac{3n\pi}{4}\right)\right).
\end{aligned}$$

We can be even more explicit by observing that

$$\sin\left(\frac{n\pi}{4}\right) + \sin\left(\frac{3n\pi}{4}\right) = \begin{cases} 0, & n \text{ even} \\ \sqrt{2}, & n = 8j+1 \text{ and } n = 8j+3 \\ -\sqrt{2}, & n = 8j+5 \text{ and } n = 8j+7 \end{cases}$$

Finally the solution can be written as

$$\begin{aligned}
u(x,t) &= \frac{2}{\pi^2}\sum_{n=1}^{+\infty}\frac{1}{n^2}\left(\sin\left(\frac{n\pi}{4}\right) + \sin\left(\frac{3n\pi}{4}\right)\right)\cos(n\pi t)\sin(n\pi x) = \\
&= \frac{2\sqrt{2}}{\pi^2}\sum_{j=0}^{+\infty}\frac{1}{(8j+1)^2}\cos((8j+1)\pi t)\sin((8j+1)\pi x) + \\
&\quad + \frac{2\sqrt{2}}{\pi^2}\sum_{j=0}^{+\infty}\frac{1}{(8j+3)^2}\cos((8j+3)\pi t)\sin((8j+3)\pi x) + \\
&\quad - \frac{2\sqrt{2}}{\pi^2}\sum_{j=0}^{+\infty}\frac{1}{(8j+5)^2}\cos((8j+5)\pi t)\sin((8j+5)\pi x) + \\
&\quad - \frac{2\sqrt{2}}{\pi^2}\sum_{j=0}^{+\infty}\frac{1}{(8j+7)^2}\cos((8j+7)\pi t)\sin((8j+7)\pi x).
\end{aligned}$$

3. Find (via Fourier series) the solution $u = u(x,t)$ of the 1-dimensional wave equation with the following boundary and initial conditions:

$$\begin{cases} u_{tt} = u_{xx}, \\ u(0,t) = 0 = u(\pi,t), & t \geqslant 0 \\ u(x,0) = 0, & 0 \leqslant x \leqslant \pi \\ u_t(x,0) = g(x), & 0 \leqslant x \leqslant \pi \end{cases}$$

where

$$g(x) = \begin{cases} \frac{x}{100}, & 0 \leqslant x \leqslant \frac{\pi}{2} \\ \frac{\pi - x}{100}, & \frac{\pi}{2} \leqslant x \leqslant \pi. \end{cases}$$

Solution:
The speed is $c = 1$ but this time the length is $L = \pi$, so that $\lambda_n = c\frac{n\pi}{L} = n$. The solution is

$$u(x,t) = \sum_{n=1}^{+\infty} (B_n \cos(nt) + B_n^* \sin(nt)) \sin(nx).$$

Imposing the initial position zero we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin(nx) = 0 \quad \Leftrightarrow \quad B_n = 0.$$

While

$$u_t(x,0) = \sum_{n=1}^{\infty} nB_n^* \sin(nx) = g(x),$$

so that nB_n^* must be the coefficients of the odd, 2π-periodic extension of g, or equivalently:

$$\begin{aligned}
B_n^* &= \frac{2}{n\pi} \int_0^{\pi} g(x) \sin(nx)\, dx = \\
&= \frac{1}{50n\pi} \int_0^{\pi/2} x \sin(nx)\, dx + \frac{1}{50n\pi} \int_{\pi/2}^{\pi} (\pi - x) \sin(nx)\, dx = \\
&= \frac{1}{50n\pi} \left(- \frac{x\cos(nx)}{n} \Big|_0^{\frac{\pi}{2}} + \frac{1}{n} \int_0^{\pi/2} \cos(nx)\, dx \right) + \frac{1}{50n\pi} \left(- \frac{(\pi - x)\cos(nx)}{n} \Big|_{\frac{\pi}{2}}^{\pi} - \frac{1}{n} \int_{\pi/2}^{\pi} \cos(nx)\, dx \right) = \\
&= - \frac{\cos\left(\frac{n\pi}{2}\right)}{100n^2} + \frac{\sin(nx)}{50n^3\pi} \Big|_0^{\frac{\pi}{2}} + \frac{\cos\left(\frac{n\pi}{2}\right)}{100n^2} - \frac{\sin(nx)}{50n^3\pi} \Big|_{\frac{\pi}{2}}^{\pi} \\
&= \frac{1}{25n^3\pi} \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n = 2j \\ \frac{(-1)^j}{25n^3\pi}, & n = 2j+1. \end{cases}
\end{aligned}$$

Finally, the solution is

$$\begin{aligned} u(x,t) &= \frac{1}{25\pi}\sum_{j=0}^{+\infty}\frac{1}{n^3}\sin\left(\frac{n\pi}{2}\right)\sin(nt)\sin(nx) \\ &= \frac{1}{25\pi}\sum_{j=0}^{+\infty}\frac{(-1)^j}{(2j+1)^3}\sin((2j+1)t)\sin((2j+1)x). \end{aligned}$$

4. Find all possible solutions of the following PDEs of the form $u(x,t) = F(x)G(t)$ (separation of variables):

a) $xu_x + u_t = 0$

Solution:
As customary, we will denote by F' the derivative of F in the variable x and by $\dot{G}$ the derivative of G in the variable t. Plugging in the equation a function of the form $u(x,t) = F(x)G(t)$ we get

$$xF'(x)G(t) + F(x)\dot{G}(t) = 0 \qquad \Leftrightarrow \qquad x\frac{F'(x)}{F(x)} = -\frac{\dot{G}(t)}{G(t)}$$

On the left-hand side we have a function of x while on the other side we have a function of t. The equality is possible only if these expressions are constant, so there must be a $\lambda \in \mathbb{R}$ such that

$$x\frac{F'(x)}{F(x)} = -\frac{\dot{G}(t)}{G(t)} = \lambda \qquad \Leftrightarrow \qquad \begin{cases} F'(x) - \frac{\lambda}{x}F(x) = 0 \\ \dot{G}(t) + \lambda G(t) = 0 \end{cases}$$

This is a system of two homogeneous, first order, ODEs, one with non-constant and the other with constant coefficients. The solutions are

$$\begin{cases} F(x) = c_1 e^{\int \frac{\lambda}{x}} = c_1 e^{\lambda \int \frac{1}{x}} = c_1\left(e^{\ln(x)}\right)^{\lambda} = c_1 x^{\lambda} \\ G(t) = c_2 e^{-\lambda t} \end{cases}$$

so that, calling $c := c_1c_2$ the product of the constants, we have solutions of the form

$$u(x,t) = F(x)G(t) = cx^{\lambda}e^{-\lambda t}, \qquad \lambda \in \mathbb{R}.$$

b) $u_x + u_t + xu = 0$

Solution:
Here we get

$$u_x + u_t + xu = F'G + F\dot{G} + xFG,$$

which we are going to impose equal to zero. We want to keep track of the variables involved in order to have clear which function depends on which variable, to then separate the equations properly:

$$\begin{aligned} &F'(x)G(t)+F(x)\dot{G}(t)+xF(x)G(t)=0 \quad \Leftrightarrow \\ \Leftrightarrow \quad &(F'(x)+xF(x))G(t)=-F(x)\dot{G}(t) \quad \Leftrightarrow \quad \frac{F'(x)}{F(x)}+x=-\frac{\dot{G}(t)}{G(t)} \end{aligned}$$

which, again, is an equality between some function of x and some other function of t, so there must be some constant $\lambda \in \mathbb{R}$ for which

$$\begin{cases} \frac{F'}{F}+x=\lambda \\ -\frac{\dot{G}}{G}=\lambda \end{cases} \Leftrightarrow \begin{cases} F'+(x-\lambda)F=0 \\ \dot{G}+\lambda G=0 \end{cases} \Leftrightarrow \begin{cases} F(x)=c_1e^{-\int(x-\lambda)}=c_1e^{-\frac{x^2}{2}+\lambda x} \\ G(t)=c_2e^{-\lambda t} \end{cases}$$

$$\rightsquigarrow \qquad u(x,t)=F(x)G(t)=ce^{-\frac{x^2}{2}+\lambda x}e^{-\lambda t}=ce^{-\frac{x^2}{2}+\lambda(x-t)}.$$

c) $t^3u_x+\cos(x)u-2u_{xt}=0$

Solution:
The equation with separated variables becomes

$$t^3F'(x)G(t)+\cos(x)F(x)G(t)-2F'(x)\dot{G}(t)=0.$$

We want to get some equation in which there are just two terms, each of which is a product of a function of x and t. So we need to gather the first term with the third term:

$$F'(x)\left(t^3G(t)-2\dot{G}(t)\right)=-\cos(x)F(x)G(t) \quad \Leftrightarrow \quad \frac{\cos(x)F(x)}{F'(x)}=\frac{2\dot{G}(t)-t^3G(t)}{G(t)}$$

Again we need to impose both terms to be constantly equal to some $\lambda \in \mathbb{R}$. In the case $\lambda \neq 0$ we can divide by it and getting the following system of ODEs:

$$\begin{cases} F'-\frac{\cos(x)}{\lambda}F=0 \\ \dot{G}-\frac{(t^3+\lambda)}{2}G=0 \end{cases} \Leftrightarrow \begin{cases} F(x)=c_1e^{\int\frac{\cos(x)}{\lambda}}=c_1e^{\frac{1}{\lambda}\int\cos(x)}=c_1e^{\frac{\sin(x)}{\lambda}} \\ G(t)=c_2e^{\int\frac{(t^3+\lambda)}{2}}=c_2e^{\frac{t^4}{8}+\frac{\lambda t}{2}}=c_2e^{\frac{t}{8}(t^3+4\lambda)} \end{cases}$$

$$\rightsquigarrow \quad \text{family of solutions:} \qquad u(x,t)=ce^{\frac{\sin(x)}{\lambda}}e^{\frac{t}{8}(t^3+4\lambda)}, \qquad \lambda \in \mathbb{R}\setminus\{0\}.$$

As one can easily observe instead the case $\lambda=0$ forces $F=0$ and so also $u=0$, which is the trivial solution and it's anyway already been considered above if the constant $c=0$.

Unimportant remark for the purpose of the exercises: what we did in each exercise is finding admissible values $\lambda \in \mathbb{R}$ for which there exists some solution of the form[1]

$$u_\lambda(x,t) = c(\lambda)F_\lambda(x)G_\lambda(t)$$

Because each of these PDEs was homogeneous, by the so-called superposition principle, then also the sum of any of these solutions is a solution.
More generally any expression of the form

$$u(x,t) = \int_{-\infty}^{+\infty} c(\lambda)F_\lambda(x)G_\lambda(t)\, d\lambda$$

will be a solution, *providing* that there are some convergence conditions (i.e. the integral converges, and it does it in such a way that this expression will be differentiable, etc. etc.).
This is in what fully consists the method of separation of variables: find the values $\lambda \in \mathbb{R}$ for which exists a specific solution with separated variables (and this is usually easy just because we have separated the variables); then 'sum' in some sense over all admissible values of λ (take a series, if it's a discrete set, take the integral, if it's continuous) these solutions to get a more general solution.
This is, more or less, one way in which every PDE is solved in what follows in the Lecture notes[2].

[1] in what follows the subscript is there to indicate a dependence on λ, it's not a derivative!
[2] you already saw the example of the Fourier series solution of the 1-dimensional wave equation