D-MAVT
D-MATL

Prof. Dr. A. Iozzi
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Analysis III

Exercise	1	2	3	4	5	Total
Value	8	10	14	8	20	60

EXAM SOLUTIONS

Stefano D'Alesio (stefano.dalesio@math.ethz.ch)

Laplace Transforms: $\big(F = \mathcal{L}(f)\big)$

	$f(t)$	$F(s)$
1)	1	$\frac{1}{s}$
2)	t	$\frac{1}{s^2}$
3)	t^2	$\frac{2}{s^3}$
4)	t^n, $n \in \mathbb{Z}_{\geqslant 0}$	$\frac{n!}{s^{n+1}}$

	$f(t)$	$F(s)$
5)	t^a, $a > 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$
6)	e^{at}	$\frac{1}{s-a}$
7)	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
8)	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$

	$f(t)$	$F(s)$
9)	$\cosh(at)$	$\frac{s}{s^2-a^2}$
t 10)	$\sinh(at)$	$\frac{a}{s^2-a^2}$
11)	$u(t-a)$	$\frac{1}{s}e^{-as}$
12)	$\delta(t-a)$	e^{-as}

(Γ = Gamma function, u = Heaviside function, δ = Delta function)

Indefinite Integrals (you may use): $\big(n \in \mathbb{Z}_{\geqslant 1}\big)$

1)	$\int x\cos(nx)\,dx = \frac{\cos(nx)+nx\sin(nx)}{n^2}$ (+ constant)
2)	$\int x^2\cos(nx)\,dx = \frac{(n^2x^2-2)\sin(nx)+2nx\cos(nx)}{n^3}$ (+ constant)
3)	$\int x\sin(nx)\,dx = \frac{\sin(nx)-nx\cos(nx)}{n^2}$ (+ constant)
4)	$\int x^2\sin(nx)\,dx = \frac{(2-n^2x^2)\cos(nx)+2nx\sin(nx)}{n^3}$ (+ constant)
5)	$\int \frac{1}{1+x^2}\,dx = \arctan(x)$ (+ constant)

1. Laplace Transform (*8 Points*)
Find, via Laplace transform, the solution of the following initial value problem:

$$\begin{array}{c} y = y(t) \\ \text{such that} \end{array} \qquad \begin{cases} y'' + 2y' + y = e^{-t} + \delta(t-3), & t \geqslant 0 \\ y(0) = 1, \\ y'(0) = -2. \end{cases} \tag{1}$$

[*Hint:* Use the pfd (= partial fraction decomposition)

$$\frac{s}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2}. \quad]$$

<u>*Solution:*</u>
We denote by $Y = Y(s)$ the Laplace transform of y. Recall the formula to transform a derivative

$$\mathcal{L}\left(y^{(n)}\right) = s^n Y - \sum_{j=0}^{n-1} s^{n-j-1} y^{(j)}(0),$$

which in the cases of our interest ($n = 1, 2$) reads as:

$$\begin{cases} \mathcal{L}(y') = sY - y(0), \\ \mathcal{L}(y'') = s^2 - sy(0) - y'(0). \end{cases}$$

The left-hand side of the ODE transforms as

$$\mathcal{L}(y'' + 2y' + y) = \mathcal{L}(y'') + \mathcal{L}(y') + \mathcal{L}(y) = s^2 Y - sy(0) - y'(0) + 2\left(sY - y(0)\right) + Y = \ldots$$

(substituting the initial conditions of the problem)

$$\cdots = s^2Y - s + 2 + 2(sY - 1) + Y = s^2Y - s + \cancel{2} + 2sY - \cancel{2} + Y = (s^2 + 2s + 1)Y - s = (s+1)^2 Y - s.$$

The right-hand side instead becomes

$$\mathcal{L}\left(e^{-t} + \delta(t-3)\right) = \mathcal{L}\left(e^{-t}\right) + \mathcal{L}\left(\delta(t-3)\right) = \frac{1}{s+1} + e^{-3s},$$

and the transformed ODE becomes the algebraic equation

$$(s+1)^2 Y - s = \frac{1}{s+1} + e^{-3s} \quad \Leftrightarrow$$

$$\Leftrightarrow \quad Y = \frac{s}{(s+1)^2} + \frac{1}{(s+1)^3} + \frac{e^{-3s}}{(s+1)^2} \overset{\text{pfd}}{=} \frac{1}{s+1} - \frac{1}{(s+1)^2} + \frac{1}{(s+1)^3} + \frac{e^{-3s}}{(s+1)^2}.$$

Consider the first three terms. These are known Laplace transforms, except they are shifted (evaluated in $s+1$ instead of s).
The s-shifting property tells us that for each real number $a \in \mathbb{R}$ and any function $g = g(t)$:

$$\mathcal{L}\left(e^{at} g\right)(s) = \mathcal{L}(g)(s-a).$$

Therefore, for each integer $k \geqslant 1$

$$\frac{1}{s^k} = \mathcal{L}\left(\frac{t^{k-1}}{(k-1)!}\right) \quad \Longrightarrow \quad \frac{1}{(s+1)^k} = \mathcal{L}\left(e^{-t}\frac{t^{k-1}}{(k-1)!}\right),$$

and the first three terms are

$$\frac{1}{s+1} - \frac{1}{(s+1)^2} + \frac{1}{(s+1)^3} = \mathcal{L}\left(e^{-t}\left(1 - t + \frac{t^2}{2}\right)\right).$$

The last term is a product of a Laplace transform with an exponential, and we can use then the t-shifting property to get

$$\frac{e^{-3s}}{(s+1)^2} = e^{-3s}\mathcal{L}\left(e^{-t}t\right) = \mathcal{L}\left(e^{-(t-3)}(t-3)u(t-3)\right).$$

Finally

$$\boxed{y = \mathcal{L}^{-1}(Y) = e^{-t}\left(1 - t + \frac{t^2}{2}\right) + e^{-(t-3)}(t-3)u(t-3).}$$

2. Wave Equation (*10 Points*)

a) (*7 Points*) Find, via d'Alembert's formula, the solution of the wave equation:

$$u = u(x,t) \text{ such that } \begin{cases} u_{tt} = c^2 u_{xx}, & t \geqslant 0,\ x \in \mathbb{R} \\ u(x,0) = e^{-x^2} + 4x + \arctan(x), & x \in \mathbb{R} \\ u_t(x,0) = -2cxe^{-x^2} + \frac{c}{1+x^2}. & x \in \mathbb{R} \end{cases} \tag{2}$$

Simplify the expression as much as possible.
[In particular, no unsolved integrals.]

<u>*Solution:*</u>
D'Alembert formula for the wave equation with initial configuration $f(x)$ and initial speed $g(x)$ is

$$u(x,t) = \frac{1}{2}\Big(\overbrace{f(x+ct) + f(x-ct)}^{\text{(I)}}\Big) + \frac{1}{2c}\overbrace{\int_{x-ct}^{x+ct} g(s)\, ds}^{\text{(II)}} = \frac{1}{2}\cdot\text{(I)} + \frac{1}{2c}\cdot\text{(II)}.$$

Let's evaluate (I) and (II) with our specific data.

$$\begin{aligned}\text{(I)} &= e^{-(x+ct)^2} + 4(x+ct) + \arctan(x+ct) + e^{-(x-ct)^2} + 4(x-ct) + \arctan(x-ct) = \\ &= e^{-(x+ct)^2} + e^{-(x-ct)^2} + 8x + \arctan(x+ct) + \arctan(x-ct).\end{aligned}$$

$$\begin{aligned}\text{(II)} &= \int_{x-ct}^{x+ct} \left(-2cse^{-s^2} + \frac{c}{1+s^2}\right) ds = c\cdot\int_{x-ct}^{x+ct}\left(-2se^{-s^2}\right) ds + c\cdot\int_{x-ct}^{x+ct}\frac{1}{1+s^2}\, ds = \\ &= c\cdot e^{-s^2}\Big|_{x-ct}^{x+ct} + c\cdot\arctan(s)\Big|_{x-ct}^{x+ct} = \\ &= c\left(e^{-(x+ct)^2} - e^{-(x-ct)^2} + \arctan(x+ct) - \arctan(x-ct)\right).\end{aligned}$$

Finally

$$\begin{aligned}u(x,t) &= \frac{1}{2}\cdot\text{(I)} + \frac{1}{2c}\cdot\text{(II)} = \\ &= \frac{1}{2}\left(e^{-(x+ct)^2} + \cancel{e^{-(x-ct)^2}} + 8x + \arctan(x+ct) + \cancel{\arctan(x-ct)}\right) + \\ &\quad + \frac{1}{2\cancel{c}}\cdot\cancel{c}\left(e^{-(x+ct)^2} - \cancel{e^{-(x-ct)^2}} + \arctan(x+ct) - \cancel{\arctan(x-ct)}\right) = \\ &= \boxed{e^{-(x+ct)^2} + 4x + \arctan(x+ct).}\end{aligned}$$

See next page!

b) (*3 Points*) Find, for each fixed $a \in \mathbb{R}$, the asymptotic limit $\lim_{t \to +\infty} u(a, t)$.

Solution:
For each fixed $a \in \mathbb{R}$, the asymptotic limit of the first addend vanishes because

$$\lim_{t \to +\infty} (a + ct)^2 = +\infty \qquad \Longrightarrow \qquad \lim_{t \to +\infty} e^{-(a+ct)^2} = 0.$$

The second addend is constantly equal to $4a$, while the third has limit

$$\lim_{t \to +\infty} \arctan(a + ct) = \frac{\pi}{2} \qquad \left(\text{because} \quad \lim_{t \to +\infty} (a + ct) = +\infty\right).$$

All three addends have limits, therefore the limit will be the sum of the three limits and it is

$$\begin{aligned}
\lim_{t \to +\infty} u(a, t) = \lim_{t \to +\infty} \left(e^{-(a+ct)^2} + 4a + \arctan(a + ct)\right) = \\
= \lim_{t \to +\infty} e^{-(a+ct)^2} + \lim_{t \to +\infty} 4a + \lim_{t \to +\infty} \arctan(a + ct) = 0 + 4a + \frac{\pi}{2} = \\
= \boxed{4a + \frac{\pi}{2}.}
\end{aligned}$$

bb) *Only if you didn't find the solution:*

You can find, in alternative, the asymptotic limit $\lim_{t \to +\infty} v(a, t)$ of

$$v(x, t) = e^{-(x+ct)^2} \sin^2(x + ct) + \frac{(x + ct)^2}{1 + (x + ct)^2}.$$

Solution:
The first term is the same term as before, multiplied by a bounded function. Therefore its asymptotic limit vanishes

$$\lim_{t \to +\infty} e^{-(a+ct)^2} \sin^2(a + ct) = 0.$$

The second term instead is a rational function with both numerator and denominator polynomials of degree 2 (in the variable t). The limit will be the ratio of the coefficients of maximum degree, which is

$$\lim_{t \to +\infty} \frac{(a + ct)^2}{1 + (a + ct)^2} = \frac{c^2}{c^2} = 1.$$

Why this is true?
When we have a quotient of polynomials of the same degree, we can extract the terms of highest degree from both and we get

$$\lim_{t\to+\infty} \frac{a_n t^n + a_{n-1}t^{n-1} + \dots a_1 t + a_0}{b_n t^n + b_{n-1}t^{n-1} + \dots + b_1 t + b_0} = \lim_{t\to+\infty} \frac{\cancel{t^n}}{\cancel{t^n}} \cdot \frac{a_n + \frac{a_{n-1}}{t} + \dots + \frac{a_1}{t^{n-1}} + \frac{a_0}{t^n}}{b_n + \frac{b_{n-1}}{t} + \dots + \frac{b_1}{t^{n-1}} + \frac{b_0}{t^n}} = \frac{a_n}{b_n}.$$

Anyway, at the end we get

$$\lim_{t\to+\infty} v(a,t) = \lim_{t\to+\infty} \left(e^{-(a+ct)^2} \sin^2(a+ct) + \frac{(a+ct)^2}{1+(a+ct)^2} \right) =$$

$$= \lim_{t\to+\infty} e^{-(a+ct)^2} \sin^2(a+ct) + \lim_{t\to+\infty} \frac{(a+ct)^2}{1+(a+ct)^2} = 0 + 1 = \boxed{1.}$$

3. **Inhomogeneous Wave Equation** (*14 Points*)
Find the solution of the following wave equation (with inhomogeneous boundary conditions) on the interval $[0,\pi]$:

$$u = u(x,t) \text{ such that } \quad \begin{cases} u_{tt} = c^2 u_{xx}, & t \geqslant 0,\ x \in [0,\pi] \\ u(0,t) = 3, & t \geqslant 0 \\ u(\pi,t) = 5, & t \geqslant 0 \\ u(x,0) = x^2 + \frac{1}{\pi}(2-\pi^2)x + 3, & x \in [0,\pi] \\ u_t(x,0) = 0. & x \in [0,\pi] \end{cases} \tag{3}$$

You must proceed as follows.

a) (*2 Points*) Find the unique function $w = w(x)$ with $w'' = 0$, $w(0) = 3$, and $w(\pi) = 5$.

Solution:
The only functions with second derivative zero are the linear functions

$$w(x) = \alpha x + \beta, \quad \alpha, \beta \in \mathbb{R}.$$

Imposing the boundary conditions we find the right coefficients

$$\begin{cases} 3 = w(0) = \alpha \cdot 0 + \beta \\ 5 = w(\pi) = \alpha \cdot \pi + \beta \end{cases} \quad \Leftrightarrow \quad \begin{cases} \alpha = \frac{2}{\pi} \\ \beta = 3 \end{cases} \quad \Leftrightarrow \quad \boxed{w(x) = \frac{2}{\pi}x + 3.}$$

b) (*4 Points*) Define $v(x,t) := u(x,t) - w(x)$. Formulate the corresponding problem for v, equivalent to (3).

Solution:
The ODE doesn't change because w is independent of time and has second derivative zero. The boundary conditions become homogeneous (that's why we chose this w)

$$v(0,t) = u(0,t) - w(0) = 3 - 3 = 0 \quad \& \quad v(\pi,t) = u(\pi,t) - w(\pi) = 5 - 5 = 0.$$

The initial position of the wave changes in

$$v(x,0) = u(x,0) - w(x) = x^2 + \frac{1}{\pi}(2-\pi^2)x + \cancel{3} - \frac{2}{\pi}x - \cancel{3} = x^2 + \cancel{\frac{2}{\pi}x} - \pi x - \cancel{\frac{2}{\pi}x} = x^2 - \pi x,$$

while the initial speed doesn't change (because, again, w is independent of time). Finally

$$\boxed{\begin{cases} v_{tt} = c^2 v_{xx}, & t \geqslant 0,\ x \in [0,\pi] \\ v(0,t) = v(\pi,t) = 0, & t \geqslant 0 \\ v(x,0) = x^2 - \pi x, & x \in [0,\pi] \\ v_t(x,0) = 0. & x \in [0,\pi] \end{cases}}$$

c) (*8 Points*)

(i) Find, using the formula from the script, the solution $v(x, t)$ of the problem you have just formulated.

<u>*Solution:*</u>
This is a standard homogeneous wave equation with homogeneous boundary conditions. The formula from the script is

$$v(x,t) = \sum_{n=1}^{+\infty} \big(B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t)\big) \sin\left(\frac{n\pi}{L}x\right), \quad \lambda_n = \frac{cn\pi}{L}$$
$$\overset{(L=\pi)}{=} \sum_{n=1}^{+\infty} \big(B_n \cos(cnt) + B_n^* \sin(cnt) \sin(nx).$$

The coefficients $B_n^* = 0$, because the initial speed is zero, while the coefficients B_n are the Fourier series coefficients of the odd, 2π-periodic extension of the initial datum $x^2 - \pi x$, that is:

$$B_n = \frac{2}{\pi}\int_0^{\pi} (x^2 - \pi x) \sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \sin(nx)\,dx - 2\int_0^{\pi} x \sin(nx)\,dx = \dots$$

[we can continue the computation using the indefinite integrals given at the beginning of the text]

$$\dots = \frac{2}{\pi}\left(\left.\frac{(2 - n^2x^2)\cos(nx) + 2nx\sin(nx)}{n^3}\right|_0^{\pi}\right) - 2\left(\left.\frac{\sin(nx) - nx\cos(nx)}{n^2}\right|_0^{\pi}\right) =$$
$$= \frac{2}{\pi}\left(\frac{(2 - n^2\pi^2)\cos(n\pi) - 2}{n^3}\right) - 2\left(-\frac{n\pi\cos(n\pi)}{n^2}\right) =$$
$$= \frac{4\cos(n\pi) - \cancel{2n^2\pi^2\cos(n\pi)} - 4 + \cancel{2n^2\pi^2\cos(n\pi)}}{\pi n^3} =$$
$$= \frac{4}{\pi n^3}(\cos(n\pi) - 1) = \frac{4}{\pi n^3}((-1)^n - 1) = -\frac{4}{\pi n^3} \cdot \begin{cases} 0 & n = 2j \\ 2 & n = 2j+1 \end{cases}$$

Finally we get the following equivalent expressions

$$\boxed{\begin{aligned} v(x,t) &= \frac{4}{\pi}\sum_{n=1}^{+\infty} \frac{((-1)^n - 1)}{n^3}\cos(cnt)\sin(nx) = \\ &= -\frac{8}{\pi}\sum_{j=0}^{+\infty} \frac{1}{(2j+1)^3}\cos\big(c(2j+1)t\big)\sin\big((2j+1)x\big). \end{aligned}}$$

(ii) Write down explicitly the solution $u(x,t)$ of the original problem (3).

Solution:
We get the following equivalent expressions

$$\boxed{\begin{aligned} u(x,t) = v(x,t) + w(x) &= \frac{4}{\pi}\left(\sum_{n=1}^{+\infty} \frac{((-1)^n - 1)}{n^3}\cos(cnt)\sin(nx)\right) + \frac{2}{\pi}x + 3 \\ &= -\frac{8}{\pi}\left(\sum_{j=0}^{+\infty} \frac{1}{(2j+1)^3}\cos\big(c(2j+1)t\big)\sin\big((2j+1)x\big)\right) + \frac{2}{\pi}x + 3. \end{aligned}}$$

4. **Laplace Equation** (*8 Points*)

Let $D_1 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leqslant 1\}$ be the unit disk centred in the origin and $\partial D_1 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$ its boundary.

Consider the Dirichlet problem (in polar coordinates):

$$\begin{matrix} u = u(r,\theta) \\ \text{such that} \end{matrix} \qquad \begin{cases} \nabla^2 u = 0, & \text{in } D_1 \\ u(1,\theta) = f(\theta), & \theta \in [0, 2\pi] \end{cases} \tag{4}$$

where

$$f(\theta) = \begin{cases} \theta e^{\theta^2}, & \theta \in [0, \pi] \\ (2\pi - \theta)e^{\pi^2+\pi-\theta}. & \theta \in [\pi, 2\pi] \end{cases}$$

Without explicitly finding the solution, answer the following questions.

(i) What is the maximum of u on the whole disk?

<u>*Solution:*</u>

By the maximum principle the maximum is on the boundary

$$\max_{D_1} u = \max_{\theta \in [0,2\pi]} f(\theta).$$

Therefore we just need to find the maximum of this function $f(\theta)$ on the interval $\theta \in [0, 2\pi]$.

A few observations are necessary.

- The function is defined by two different formulas in the two subintervals

$$f(\theta) = \begin{cases} f_1(\theta) = \theta e^{\theta^2}, & \theta \in [0, \pi] \\ f_2(\theta) = (2\pi - \theta)e^{\pi^2+\pi-\theta}. & \theta \in [\pi, 2\pi] \end{cases}$$

  so it's convenient to study these functions separately and then compare their maximums.

- The function $f_1(\theta)$ is strictly increasing, while the function $f_2(\theta)$ is strictly decreasing. This can be either observed directly[1], or computing the derivatives

$$\begin{cases} f_1'(\theta) = (1 + 2\theta^2)e^{\theta^2} > 0, & \theta \in [0, \pi] \\ f_2'(\theta) = (\theta - 2\pi - 1)e^{\pi^2+\pi-\theta} < 0. & \theta \in [\pi, 2\pi] \end{cases}$$

It follows that the maximum of $f_1(\theta)$ is at $\theta = \pi$, and same for $f_2(\theta)$. Of course these values will coincide because $\theta = \pi$ is the conjunction point between the two subintervals,

$$f_1(\pi) = \pi e^{\pi^2} = f_2(\pi),$$

and this is the global maximum:

$$\boxed{\max_{D_1} u = \pi e^{\pi^2}.}$$

[1] f_1 is a product of nonnegative, increasing functions while f_2 is a product of nonnegative, decreasing functions.

(i) Same question for the minimum.

<u>*Solution:*</u>
As for the maximum principle, there is also a minimum principle, because

$$\min_{D_1} u = -\max_{D_1}(-u) \overset{(\star)}{=} -\max_{\partial D_1}(-u) = \min_{\partial D_1} u,$$

where $(\star)$ is true because also $-u$ is harmonic.
So, we need to find the minimum of the function $f(\theta)$.
For the same observations we made before it's clear that this is reached in $\theta = 0$ (if we look it from the first subinterval) or $\theta = 2\pi$ (if we look it from the second subinterval), with value

$$f_1(0) = 0 = f_2(2\pi).$$

Therefore the answer is

$$\boxed{\min_{D_1} u = 0.}$$

5. Heat Equation (*20 Points*)

a) (*10 Points*) Find, via separation of variables, the general solution of the heat equation (with homogeneous boundary conditions) on the interval $[0,\pi]$ - with initial condition on the derivative:

$$u = u(x,t) \text{ such that } \quad \begin{cases} u_t = c^2 u_{xx}, & t \geqslant 0,\, x \in [0,\pi] \\ u(0,t) = u(\pi,t) = 0, & t \geqslant 0 \\ u_x(x,0) = h(x), & x \in [0,\pi] \end{cases} \tag{5}$$

where $h(x)$ is a function with $\int_0^\pi h(x)\,dx = 0$.
Show all the steps of the method of separation of variables.
[*Remark:* The solution will be a series. The coefficients must be written in terms of the initial datum $h(x)$.]

Solution:
We separate variables $u(x,t) = F(x)G(t)$ and the ODE becomes

$$F\dot{G} = c^2 F'' G \quad \Leftrightarrow \quad \frac{\dot{G}}{c^2 G} = \frac{F''}{F} = k,$$

for some real constant $k \in \mathbb{R}$.
The homogeneous boundary conditions ($u(0,t) = u(\pi,t) = 0$) force F to be zero on the boundary, otherwise G should be identically zero for each time, and therefore also u would be. In other words we need to find

$$F = F(x) \quad \text{s.t.} \quad \begin{cases} F'' = kF, \\ F(0) = F(\pi) = 0. \end{cases}$$

Let's distinguish the possible cases for k.
If $k > 0$ then the general solution of the ODE is

$$F(x) = \lambda_1 e^{\sqrt{k}x} + \lambda_2 e^{-\sqrt{k}x}, \quad \lambda_1, \lambda_2 \in \mathbb{R},$$

and the boundary conditions become

$$\begin{cases} \lambda_1 + \lambda_2 = 0 \\ \lambda_1 e^{\sqrt{k}\pi} + \lambda_2 e^{-\sqrt{k}\pi} = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \lambda_2 = -\lambda_1 \\ \lambda_1 \left(e^{\sqrt{k}\pi} - e^{-\sqrt{k}\pi}\right) = 0 \end{cases} \quad \Leftrightarrow \quad \lambda_1 = \lambda_2 = 0,$$

so the solution is trivial.
If $k = 0$ the general solution is

$$F(x) = \lambda_1 x + \lambda_2, \quad \lambda_1, \lambda_2 \in \mathbb{R},$$

and again the boundary conditions force the solution to be zero

$$\begin{cases} \lambda_2 = 0 \\ \lambda_1 \pi = 0 \end{cases} \quad \Leftrightarrow \quad \lambda_1 = \lambda_2 = 0.$$

If $k < 0$ it's convenient to write it in the form $k = -p^2$ (for a unique $p > 0$) and the ODE $F'' = -p^2 F$ has general solution

$$F(x) = A\cos(px) + C\sin(px).$$

The first boundary condition forces $A = 0$ while the second boundary condition is actually a condition on the parameter p, in fact

$$0 = F(\pi) = C\sin(p\pi) \quad \Leftrightarrow \quad p\pi = n\pi,\ n \in \mathbb{Z}_{\geqslant 1} \quad \Leftrightarrow \quad p = n,\ n \in \mathbb{Z}_{\geqslant 1}.$$

So, for each $n \in \mathbb{Z}_{\geqslant 1}$ we have a solution $F_n(x) = C_n \sin(nx)$. The ODE for the time-dependent function becomes

$$\dot{G} = kc^2 G = -p^2 c^2 G = -n^2 c^2 G = -\lambda_n^2 G, \quad (\lambda_n := cn)$$

which has unique solution $G_n(t) = D_n e^{-\lambda_n^2 t}$. Putting together the constants in a unique constant $B_n = C_n D_n$ we have solutions

$$u_n(x,t) = F_n(x) G_n(t) = B_n \sin(nx)\, e^{-\lambda_n^2 t},$$

and general solution, by the superposition principle,

$$\boxed{u(x,t) = \sum_{n=1}^{+\infty} u_n(x,t) = \sum_{n=1}^{+\infty} B_n \sin(nx)\, e^{-\lambda_n^2 t}.}$$

Now we have to find the coefficients B_n by imposing the initial condition. First we derive term by term the series, and we get

$$u_x(x,t) = \sum_{n=1}^{+\infty} B_n n \cos(nx)\, e^{-\lambda_n^2 t}.$$

Then we impose at time zero

$$h(x) = u_x(x,0) = \sum_{n=1}^{+\infty} B_n n \cos(nx)\, e^{-\lambda_n^2 t}\Big|_{t=0} = \sum_{n=1}^{+\infty} B_n n \cos(nx).$$

The Fourier series of the even, 2π-periodic extension, $h_{\text{even}}(x)$, is made only of cosines

$$h_{\text{even}}(x) = a_0 + \sum_{n=1}^{+\infty} a_n \cos(nx) = \sum_{n=1}^{+\infty} a_n \cos(nx).$$

(We observed that the constant term vanishes because $a_0 = \frac{1}{\pi}\int_0^{\pi} h(x)\,dx = 0$). Therefore, comparing the two expressions above

$$\sum_{n=1}^{+\infty} B_n n \cos(nx) = \sum_{n=1}^{+\infty} a_n \cos(nx),$$

we get

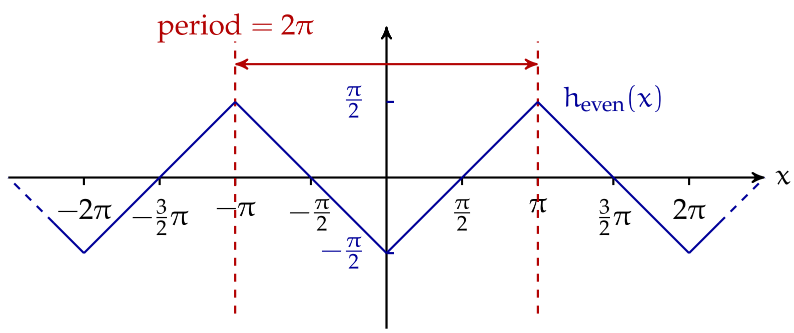
$$B_n n = a_n = \underset{2\pi\text{-periodic extension, } h_{\text{even}}(x)}{\text{coefficients of the even,}} = \frac{2}{\pi}\int_0^{\pi} h(x)\cos(nx)\,dx,$$

or

$$\boxed{B_n = \frac{2}{n\pi}\int_0^{\pi} h(x)\cos(nx)\,dx.}$$

b) (*8 Points*) Consider the function $h(x) = x - \frac{\pi}{2}$, on the interval $x \in [0, \pi]$, and let $h_{\text{even}}(x)$ be its even, 2π-periodic extension.
Sketch the graph of $h_{\text{even}}(x)$ and find its Fourier series.
[To get full points sketch the graph at least in the interval $x \in [-2\pi, 2\pi]$.]

<u>*Solution:*</u>



The even extension will have Fourier series of the form

$$a_0 + \sum_{n=1}^{+\infty} a_n\cos(nx), \qquad \begin{cases} a_0 = \frac{1}{\pi}\int_0^{\pi} g(x)\,dx \\ a_n = \frac{2}{\pi}\int_0^{\pi} g(x)\,dx \quad (n \geqslant 1)\end{cases}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} g(x)\,dx = \frac{1}{\pi}\int_0^{\pi}\left(x - \frac{\pi}{2}\right)dx = \frac{1}{\pi}\cdot\left(\frac{1}{2}x^2 - \frac{\pi}{2}x\right)\Bigg|_0^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right) = 0,$$

(as one could also observe by the graph of $g(x)$). For the other coefficients we can use the primitive we have at the beginning of the text

$$a_n = \frac{2}{\pi}\int_0^{\pi} g(x)\cos(nx),\,dx = \frac{2}{\pi}\int_0^{\pi}\left(x - \frac{\pi}{2}\right)\cos(nx)\,dx =$$

$$= \frac{2}{\pi}\cdot\frac{\cos(nx) + nx\sin(nx)}{n^2}\Bigg|_0^{\pi} - \cancel{\frac{\sin(nx)}{n}\Bigg|_0^{\pi}} = \frac{2}{\pi n^2}\cdot(\cos(n\pi) - 1) =$$

$$= \frac{2}{\pi n^2}\left((-1)^n - 1\right) = \begin{cases} 0, & n = 2j \\ -\frac{4}{\pi n^2}. & n = 2j+1\end{cases}$$

The Fourier series is then given by the two equivalent expressions

$$\boxed{\frac{2}{\pi}\sum_{n=1}^{+\infty}\frac{\big((-1)^n-1\big)}{n^2}\cos(nx) = -\frac{4}{\pi}\sum_{j=0}^{+\infty}\frac{1}{(2j+1)^2}\cos\big((2j+1)x\big).}$$

c) (*2 Points*) Write down explicitly the solution $u(x,t)$ of the problem (5) with the function $h(x)$ of subtask **5.b)**.

Solution:
With the same notations as before

$$B_n = \frac{1}{n}a_n = \frac{2}{\pi n^3}\big((-1)^n-1\big) = \begin{cases} 0, & n = 2j \\ -\frac{4}{\pi n^3}. & n = 2j+1 \end{cases}$$

Finally we get the two equivalent expressions

$$\boxed{\begin{aligned} u(x,t) &= \frac{2}{\pi}\sum_{n=1}^{+\infty}\frac{\big((-1)^n-1\big)}{n^3}\sin(nx)e^{-c^2n^2t} = \\ &= -\frac{4}{\pi}\sum_{j=0}^{+\infty}\frac{1}{(2j+1)^3}\sin\big((2j+1)x\big)e^{-c^2(2j+1)^2t}. \end{aligned}}$$