ETH Zürich
Prof. Dr. A. Iozzi

Winter 2019
24 January

Exam Analysis III
D–MAVT, D–MATL

Please fill!

Surname:	
First Name:	
Student Card Nr.:	

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Exercise	Value	Points	Control
1	8		
2	10		
3	14		
4	8		
5	20		
Total	60		

Please do not fill!

Completeness	

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Important: Before the exam starts, please

- Turn off any mobile phone/device and place it inside your bag.
- Put your bags on the floor. No bags on the desk!
- Place your Student Card (**Legi**) on the desk.
- **Fill in** the front page of the exam **with your generalities**.

During the exam, please

- Start every exercise on a new piece of paper.
- Put your name on the top right corner of every page.
- You are expected to motivate your answers. Please write down calculations and intermediate results.
- Provide at most **one** solution to each exercise.
- **Do not** write with **pencils**. **Do not** use any **red** or **green** ink pens.

Allowed aids:

- 20 pages (= 10 sheets) DIN A4 handwritten or typed personal summary.
- An English (or English-German) dictionary.

Not allowed:

No further aids are allowed. In particular neither communication devices, nor pocket calculators.

Advice: Have a look at the whole exam.
Most exercises are made of multiple subtasks.
The exam is written in such a way you can often solve a subtask even if you didn't solve the previous ones.

Good Luck!

See next page!

Laplace Transforms: $\big(F = \mathcal{L}(f)\big)$

	$f(t)$	$F(s)$
1)	1	$\frac{1}{s}$
2)	t	$\frac{1}{s^2}$
3)	t^2	$\frac{2}{s^3}$
4)	$t^n,\ n \in \mathbb{Z}_{\geqslant 0}$	$\frac{n!}{s^{n+1}}$

	$f(t)$	$F(s)$
5)	$t^a,\ a > 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$
6)	e^{at}	$\frac{1}{s-a}$
7)	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
8)	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$

	$f(t)$	$F(s)$
9)	$\cosh(at)$	$\frac{s}{s^2-a^2}$
10)	$\sinh(at)$	$\frac{a}{s^2-a^2}$
11)	$u(t-a)$	$\frac{1}{s}e^{-as}$
12)	$\delta(t-a)$	e^{-as}

(Γ = Gamma function, u = Heaviside function, δ = Delta function)

Indefinite Integrals (you may use): $\big(n \in \mathbb{Z}_{\geqslant 1}\big)$

1)	$\displaystyle\int x\cos(nx)\,dx = \frac{\cos(nx) + nx\sin(nx)}{n^2}$ (+ constant)
2)	$\displaystyle\int x^2\cos(nx)\,dx = \frac{(n^2x^2-2)\sin(nx) + 2nx\cos(nx)}{n^3}$ (+ constant)
3)	$\displaystyle\int x\sin(nx)\,dx = \frac{\sin(nx) - nx\cos(nx)}{n^2}$ (+ constant)
4)	$\displaystyle\int x^2\sin(nx)\,dx = \frac{(2-n^2x^2)\cos(nx) + 2nx\sin(nx)}{n^3}$ (+ constant)
5)	$\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan(x)$ (+ constant)

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1. Laplace Transform (*8 Points*)

Find, via Laplace transform, the solution of the following initial value problem:

$$y = y(t) \text{ such that } \quad \begin{cases} y'' + 2y' + y = e^{-t} + \delta(t-3), & t \geqslant 0 \\ y(0) = 1, \\ y'(0) = -2. \end{cases} \qquad (1)$$

[*Hint:* Use the pfd (= partial fraction decomposition)

$$\frac{s}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2}.\ ]$$

2. Wave Equation (*10 Points*)

a) (*7 Points*) Find, via d'Alembert's formula, the solution of the wave equation:

$$u = u(x,t) \text{ such that } \begin{cases} u_{tt} = c^2 u_{xx}, & t \geqslant 0,\ x \in \mathbb{R} \\ u(x,0) = e^{-x^2} + 4x + \arctan(x), & x \in \mathbb{R} \\ u_t(x,0) = -2cxe^{-x^2} + \frac{c}{1+x^2}. & x \in \mathbb{R} \end{cases} \tag{2}$$

Simplify the expression as much as possible.
[In particular, no unsolved integrals.]

b) (*3 Points*) Find, for each fixed $a \in \mathbb{R}$, the asymptotic limit $\lim\limits_{t\to+\infty} u(a,t)$.

bb) *Only if you didn't find the solution:*

You can find, in alternative, the asymptotic limit $\lim\limits_{t\to+\infty} v(a,t)$ of

$$v(x,t) = e^{-(x+ct)^2} \sin^2(x+ct) + \frac{(x+ct)^2}{1+(x+ct)^2}.$$

3. Inhomogeneous Wave Equation (*14 Points*)
Find the solution of the following wave equation (with inhomogeneous boundary conditions) on the interval $[0,\pi]$:

$$u = u(x,t) \text{ such that } \begin{cases} u_{tt} = c^2 u_{xx}, & t \geqslant 0,\ x \in [0,\pi] \\ u(0,t) = 3, & t \geqslant 0 \\ u(\pi,t) = 5, & t \geqslant 0 \\ u(x,0) = x^2 + \frac{1}{\pi}(2-\pi^2)x + 3, & x \in [0,\pi] \\ u_t(x,0) = 0. & x \in [0,\pi] \end{cases} \tag{3}$$

You must proceed as follows.

a) (*2 Points*) Find the unique function $w = w(x)$ with $w'' = 0$, $w(0) = 3$, and $w(\pi) = 5$.

b) (*4 Points*) Define $v(x,t) := u(x,t) - w(x)$. Formulate the corresponding problem for v, equivalent to (3).

c) (*8 Points*)

(i) Find, using the formula from the script, the solution $v(x,t)$ of the problem you have just formulated.

(ii) Write down explicitly the solution $u(x,t)$ of the original problem (3).

4. Laplace Equation (*8 Points*)

Let $D_1 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leqslant 1\}$ be the unit disk centred in the origin and $\partial D_1 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$ its boundary.

Consider the Dirichlet problem (in polar coordinates):

$$\begin{array}{c} u = u(r,\theta) \\ \text{such that} \end{array} \qquad \begin{cases} \nabla^2 u = 0, & \text{in } D_1 \\ u(1,\theta) = f(\theta), & \theta \in [0, 2\pi] \end{cases} \tag{4}$$

where

$$f(\theta) = \begin{cases} \theta e^{\theta^2}, & \theta \in [0,\pi] \\ (2\pi - \theta)e^{\pi^2+\pi-\theta}. & \theta \in [\pi, 2\pi] \end{cases}$$

Without explicitly finding the solution, answer the following questions.

(i) What is the maximum of u on the whole disk?

(ii) Same question for the minimum.

5. Heat Equation (*20 Points*)

a) (*10 Points*) Find, via separation of variables, the general solution of the heat equation (with homogeneous boundary conditions) on the interval $[0, \pi]$ - with initial condition on the derivative:

$$\begin{array}{c} u = u(x,t) \\ \text{such that} \end{array} \qquad \begin{cases} u_t = c^2 u_{xx}, & t \geqslant 0,\ x \in [0,\pi] \\ u(0,t) = u(\pi,t) = 0, & t \geqslant 0 \\ u_x(x,0) = h(x), & x \in [0,\pi] \end{cases} \tag{5}$$

where $h(x)$ is a function with $\int_0^\pi h(x)\,dx = 0$.

Show all the steps of the method of separation of variables.

[*Remark:* The solution will be a series. The coefficients must be written in terms of the initial datum $h(x)$.]

b) (*8 Points*) Consider the function $h(x) = x - \frac{\pi}{2}$, on the interval $x \in [0, \pi]$, and let $h_{\text{even}}(x)$ be its even, 2π-periodic extension.

Sketch the graph of $h_{\text{even}}(x)$ and find its Fourier series.

[To get full points sketch the graph at least in the interval $x \in [-2\pi, 2\pi]$.]

c) (*2 Points*) Write down explicitly the solution $u(x,t)$ of the problem (5) with the function $h(x)$ of subtask **5.b)**.