Answer sheet compilation instructions

- Use only black or blue pen.

For open answers:

- Write **clearly** only inside the provided boxes.
- Each box should contain a single integer (positive or negative) or a single fraction (reduced to lowest form).

For multiple choice questions:

- Fill the circle of the answer you consider correct (only one answer is correct).
- Remarks and computations have **no** influence on points awarded.
- Any unclear or double marks will be counted as answer not given (0 points).
- Wrong answers give **negative** points.

Exam instructions

- Turn off your devices and leave them in your bag.
- Only pens and Legi should be on the table.
- Fill your last name and Legi number on the answer sheet.
- Turn this sheet only when instructed to do so.
- At the end of the exam, give the single answer sheet which you want to submit to an assistant, and take everything else with you.

Questions

NumCSE endterm, HS 2018

1. *Fourier transform [5 points].*

   (a) Which of the functions in the figures

Figure 1:

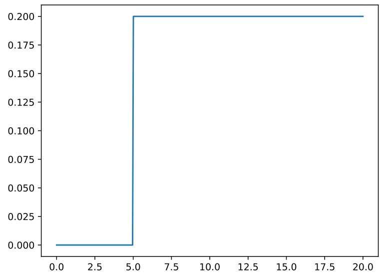


Figure 2:

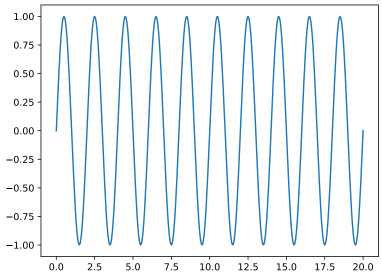
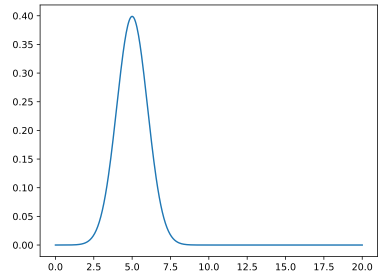


Figure 3:



has the following Fourier transform (absolute values of the Fourier coefficients are plotted):

Solution: Figure 2 [1,0,-1]

   (b) Consider the vectors

$$a := (1, 1, 2, 0) \quad \text{and} \quad b := (2, 1, 3, 2),$$

their 4-periodic convolution $c := a *_4 b$ and the Fourier matrix

$$\mathbf{F}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}.$$

Compute the discrete Fourier transform of c.

Solution: $\mathrm{DFT}(c) = (\dots, 32, 2, 4, 2, \dots)$ **[2 if all correct and 0 otherwise.]**

(c) Consider the `C++/Eigen` code

```cpp
Eigen::VectorXcd dft_1(const Eigen::VectorXcd &y) {
    const double pi = 3.14159265359;
    const int n = y.size();
    const int n_half = n / 2;

    if (n == 1) return y;

    Eigen::VectorXcd y1(n_half);
    Eigen::VectorXcd y2(n_half);

    for (int i = 0; i < n_half; ++i) {
        y1(i) = y(2 * i);
        y2(i) = y(2 * i + 1);
    }

    const Eigen::VectorXcd c1 = dft_1(y1);
    const Eigen::VectorXcd c2 = dft_1(y2);

    const std::complex<double> omega = std::exp(-2.0 * pi / n * std::complex<double>(0.0, 1.0));
    std::complex<double> omega_k(1.0,0.0);
    Eigen::VectorXcd c(n);

    for (int k = 0; k < n; ++k) {
        c(k) = c1(k % n_half) + c2(k % n_half) * omega_k;
        omega_k *= omega;
    }

    return c;
}

Eigen::VectorXcd dft_2(const Eigen::VectorXcd &y) {
    const double pi = 3.14159265359;
    const int n = y.size();

    Eigen::MatrixXcd F(n, n);

    for (int j = 0; j < n; ++j) {
        for (int i = 0; i < n; ++i) {
            F(i, j) = std::exp(-2.0 * pi / n * i * j * std::complex<double>(0.0, 1.0));
        }
    }

    return F * y;
}
```

./codes/complexityA.cpp

You may assume that the length $n \geq 2$ of the vector `y` is a power of 2. The two functions `dft_1` and `dft_2` perform the same task, but with a different algorithm. Provide the lowest numbers $p, q, \alpha, \beta \in \mathbb{N}_0$ such that the asymptotic complexities for large n are given by:

- `dft_1`: $\mathcal{O}(n^p \log^q(n))$
- `dft_2`: $\mathcal{O}(n^\alpha \log^\beta(n))$

Solution: $p = q = 1$ **and** $\alpha = 2, \beta = 0$. **[+1 for every correct pair.]**

2. *Chebyshev interpolation [3 points].*

   Let $\mathsf{L}_{\mathcal{T}}f$ denote the polynomial interpolant of the function $f : I \to \mathbb{R}$ for the set of Chebyshev nodes $\mathcal{T} := \{t_0, t_1, \ldots, t_n\}$, $n \in \mathbb{N}$. What kind of convergence, with respect to n, is to be expected for the approximation error of the Chebyshev interpolant ($\|f - \mathsf{L}_{\mathcal{T}}f\|_{L^\infty(I)}$) in each of the following cases?

   (a) $f \in C^\infty(I)$
   Solution: Exponential convergence Points: [+1,0,-1]

   (b) $f \in C^2(I)$, but $f \notin C^3(I)$
   Solution: Algebraic convergence Points: [+1,0,-1]

   (c) $f \in C^0(I)$, but $f \notin C^1(I)$
   Solution: Algebraic convergence Points: [+1,0,-1]

3. *Convergence of Gauss-Legendre Quadrature Formula [3 points].*

   Let $Q_n(f)$ be the n-point **Gauss-Legendre** quadrature rule on $\Omega \subset \mathbb{R}$ for an integrand f. What kind of convergence, with respect to n, is to be expected for the quadrature error $E_n(f) := \left|\int_\Omega f(t)\,\mathrm{d}t - Q_n(f)\right|$ in each of the following cases:

   (a) $f(t) = t^{\frac{5}{2}}$ and $\Omega = [0, 1]$
   Solution: Algebraic convergence Points: [+1,0,-1]
   For $f \in C^r(\Omega)$, Gauss-Legendre quadrature formula converges algebraically with $\mathcal{O}(n^{-r})$. Here $f \in C^2(\Omega)$.

   (b) $f \in C^\infty(\Omega)$
   Solution: Exponential convergence Points: [+1,0,-1]
   For a smooth function, Gauss-Legendre quadrature formula converges exponentially with $\mathcal{O}(\lambda^n)$, $\lambda \in (0, 1)$.

   (c) $f(t) = |t|$ and $\Omega = [-1, 1]$
   Solution: Algebraic convergence Points: [+1,0,-1]
   This is because $f \in C^0(\Omega)$.

4. *Convergence of Composite Simpson rule [3 points].*

   Let $Q_n(f)$ be the n-point **Composite Simpson** quadrature rule with equally spaced nodes on $\Omega \subset \mathbb{R}$ for an integrand f. What kind of convergence, with respect to n, is to be expected for the quadrature error $E_n(f) := \left|\int_\Omega f(t)\,\mathrm{d}t - Q_n(f)\right|$ in each of the following cases:

   (a) $f(t) = t^{\frac{5}{2}}$ and $\Omega = [0, 1]$
   Solution: Algebraic convergence Points: [+1,0,-1]
   For $f \in C^r(\Omega)$, Composite quadrature formula (with local order q) converges algebraically with $\mathcal{O}(n^{-\min\{r,q\}})$. Here $f \in C^2(\Omega)$.

   (b) $f(t) = t^{\frac{3}{2}}$ and $\Omega = [1, 2]$
   Solution: Algebraic convergence Points: [+1,0,-1]
   Here $f \in C^\infty(\Omega)$. Regardless, Composite quadrature formula (with local order q) converges algebraically with $\mathcal{O}(n^{-q})$.

   (c) $f(t) = |t|$ and $\Omega = [-1, 1]$
   Solution: Algebraic convergence Points: [+1,0,-1]
   This is because $f \in C^0(\Omega)$.

5. *Runge-Kutta methods [5 points].*

(a) Consider the following 2-stage Runge-Kutta method used for solving the ODE $\dot{y} = f(y)$:

$$y_{k+1} = y_k - \frac{h}{2}(k_1 - 3k_2)$$

where

$$k_1 = f\left(t_k + \frac{h}{2},\ y_k + \frac{h}{2}k_1\right),$$
$$k_2 = f\left(t_k + \frac{3h}{2},\ y_k - \frac{h}{2}(k_1 - 4hk_2)\right).$$

Complete the entries of the Butcher tableau corresponding to this method.

Solution: (+2 if all entries are correct, no negatives)

There was a typo in this subproblem. The term marked in red should have been $-4k_2$ instead of $-4hk_2$.

The correct Butcher tableau is:

$$\begin{array}{c|cc} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{2} & -\frac{1}{2} & \mathbf{2} \\ \hline & -\frac{1}{2} & \frac{3}{2} \end{array}$$

The entry in blue is the one affected by the typo. During correction of this subproblem, the entry corresponding to this box (i.e a_{22}) has been disregarded. Therefore, 2 points have been awarded for this subproblem if all the other entries except a_{22} were correct.

(b) Consider the following Butcher tableau for a Runge-Kutta method:

$$\begin{array}{c|cc} \frac{1}{2} + \frac{\sqrt{3}}{6} & \frac{1}{2} + \frac{\sqrt{3}}{6} & 0 \\ \frac{1}{2} - \frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{3} & \frac{1}{2} + \frac{\sqrt{3}}{6} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Does this correspond to an explicit or an implicit method?

Solution: Implicit method. Points: [+1,0,-1]

This is because it does not have a strictly lower triangular matrix. In fact, this is Crouzeix's two-stage, 3rd order Diagonally Implicit Runge Kutta method.

(c) Consider the following Butcher tableau for a Runge-Kutta method:

$$\begin{array}{c|cccc} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{2}{3} & \gamma & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ \hline & \delta & -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \end{array}$$

What should be the values of γ and δ such that the method is consistent?

Solution: (+1 for correct γ, +1 for correct δ, no negatives)

$$\gamma = \frac{1}{6} \quad \textbf{and} \quad \delta = \frac{3}{2}.$$

For consistency, the following conditions need to be satisfied:

$$\sum_{i=1}^{s} b_i = 1$$

and

$$c_i = \sum_{j=1}^{s} a_{ij}.$$

6. *Stiffness and stability [4 points].*

(a) Consider the second order, scalar ODE

$$\ddot{y}(t) = -y(t)$$

and its equivalent system of first order ODEs

$$\dot{\mathbf{z}}(t) = \mathbf{A}\,\mathbf{z}(t), \tag{1}$$

where $t \geq 0, \mathbf{z}(0) = (y(0), \dot{y}(0))^\top$ and $\mathbf{A} \in \mathbb{R}^{2\times 2}$. Compute explicitly the entries of $\mathbf{A}$.
Solution: [1 if all correct, 0 otherwise]

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(b) Is the ODE (1) stiff (where $\mathbf{A}$ is the solution from Part (a))?
Solution: no [1, 0, -1]

(c) Is the following ODE stiff for $c \gg 1$?

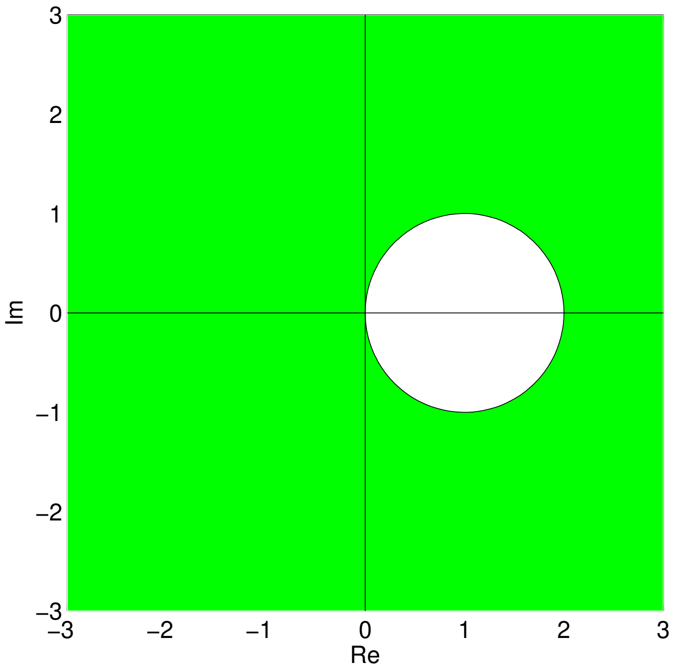
$$\dot{\mathbf{y}}(t) = \begin{pmatrix} -c & 1 \\ -1 & -c \end{pmatrix} \mathbf{y}(t)$$

Solution: yes [1, 0, -1]

(d) Which of the methods

(i) explicit midpoint
(ii) explicit Euler
(iii) implicit Euler

has the following stability region (shaded grey area):



Solution: (iii) [1, 0, -1]