Answer sheet compilation instructions

- Use only black or blue pen.

For open answers:

- Write **clearly** only inside the provided boxes.
- Each box should contain a single integer (positive or negative).

For multiple choice questions:

- Fill the circle of the answer you consider correct (only one answer is correct).
- Remarks and computations have **no** influence on points awarded.
- Any unclear or double marks will be counted as answer not given (0 points).
- Wrong answers give **negative** points.

Exam instructions

- Turn off your devices and leave them in your bag.
- Only pens and Legi should be on the table.
- Fill last name and Legi number on the answer sheet.
- Turn this sheet only when instructed to do so.
- At the end of the exam, give the single answer sheet which you want to submit to an assistant, and take everything else with you.

Questions

NumCSE midterm, HS 2018

1. *Convolution [6 points].*

   Let

   $$x = (3, 6, 12, 24, 9, 9, 9, 9, 9, 9, -12, 11, -12, 14).$$

   Let $*$ indicate the discrete linear convolution and $*_p$ the discrete periodic convolution with period p. Let $z[i]$ indicate the element at position i of any vector z. For instance: $x[0] = 3$ and $x[12] = 14$.

   (a) If $y = (1, 2, 3, 4, 5)$, what is the length of $x * y$? **17 [1 for correct answer,0 for answer not given,0 for wrong answer]**

   (b) If $y = (2, -2, 2)$ what is $(y *_3 y)[0]$? **-4 [1,0,0]**

   (c) If $y = (1, -1, 1, -1, 1)$, what is $(x * y)[3]$? **15 [1,0,0]**

   (d) If $y = (7, 7, -7, -7, 7, 7, -7, -7)$ what is $(x * y)[11]$? **14 [1,0,0]**

   (e) If $y = (2, -1)$, what is $(x *_{16} y)[2018]$? **18 [2,0,0]**

2. *Singular Value Decomposition [4 points].*

Let $A \in \mathbb{R}^{3,2}$ be defined as

$$A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

(a) What are the non-zero singular values of A? **2,1 [1 for both correct,0,0]**

(b) Consider the *full* singular value decomposition $A = U\Sigma V^\top$ of A. Determine $a, b, \alpha, \beta \in \mathbb{N}$ such that $U \in \mathbb{R}^{a,b}$ and $V \in \mathbb{R}^{\alpha,\beta}$. $U \in \mathbb{R}^{3,3}, V \in \mathbb{R}^{2,2}$ **[1 for all correct,0,0]**

(c) Consider the *reduced* singular value decomposition $A = \tilde{U}\tilde{\Sigma}\tilde{V}^\top$ of A. Determine $a, b, \alpha, \beta \in \mathbb{N}$ such that $\tilde{U} \in \mathbb{R}^{a,b}$ and $\tilde{V} \in \mathbb{R}^{\alpha,\beta}$. $\tilde{U} \in \mathbb{R}^{3,2}, \tilde{V} \in \mathbb{R}^{2,2}$ **[1 for all correct,0,0]**

(d) Let $\tilde{A} \in \mathbb{R}^{3,2}$ be the best rank–1 approximation of A. Let $\|\cdot\|_F$ denote the Frobenius norm. What is the value of $\|A - \tilde{A}\|_F$? **1 [1,0,0]**

3. *Complexity [4 points].*

Consider the following `Eigen/C++` code:

```
8     MatrixXd A = MatrixXd::Zero(n, n);
9
10    A(0, 0) = 1.0; A(1, 0) = 1.0;
11    for (int j = 1; j < n - 1; ++j) {
12        for (int i = j - 1; i < j + 2; ++i) {
13            A(i, j) = 1.0;
14        }
15    }
16    A(n - 2, n - 1) = 1.0; A(n - 1, n - 1) = 1.0;
17
18    MatrixXd Q = A.householderQr().householderQ();
19    cout << Q;
20
21    for (int i = 0; i < n*n; ++i) {
22        VectorXd b = VectorXd::Random(n);
23        VectorXd M = A.fullPivLu().solve(b);
24        cout << M;
25    }
26
27    FullPivLU<MatrixXd> lu = A.fullPivLu();
28    for (int i = 0; i < n*n; ++i) {
29        VectorXd b = VectorXd::Random(n);
30        VectorXd M = lu.solve(b);
31        cout << M;
32    }
```

complexity.cpp

You can assume that lines 22 and 29 run in $O(\texttt{n})$. What integer α is such that $O(\texttt{n}^\alpha)$ is the lowest correct asymptotic complexity of...

(a) ...lines 10–16? **1 [1,0,0]**

(b) ...lines 18–19? **3 [1,0,0]**

(c) ...lines 21–25? **5 [1,0,0]**

(d) ...lines 27–32? **4 [1,0,0]**

4. *Cancellation [4 points].*

Which side of the equations below should be preferred in order to minimize the impact of cancellation? On the answer sheet, we abbreviate left-hand side (LHS) and right-hand side (RHS).

(a) For large $x \gg 1$:

$$\frac{1}{\sqrt{x^2+1}+x} = \sqrt{x^2+1} - x$$

LHS [1,0,-1]

(b) For small $x > 0$:

$$(1-x)^2 - 1 = x^2 - 2x$$

RHS [1,0,-1]

(c) For large $x \gg 1$:

$$\frac{(x+1)^2 - x^2}{x} = 2 + \frac{1}{x}$$

RHS [1,0,-1]

(d) For small $x > 0$:

$$\frac{2x^2}{(1+2x)(1+x)} = \frac{1}{1+2x} - \frac{1-x}{1+x}$$

LHS [1,0,-1]

5. *Householder reflections [8 points].*

The Householder matrix for a reflection about the hyper-plane with normal vector $\mathbf{v}$ is

$$\mathbf{H_v} := \mathbf{I_m} - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}} = \mathbf{I_m} - 2\tilde{\mathbf{v}}\tilde{\mathbf{v}}^T ,$$

where $\tilde{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|_2}$ is a unit vector and $\mathbf{I_m}$ is an identity matrix of size $m \times m$. Note that $\mathbf{H_v}$ is symmetric and orthogonal.

We want to reduce a matrix $\mathbf{A} \in \mathbb{R}^{3,3}$ to an upper triangular form $\mathbf{R}$ using successive Householder transformations

$$\mathbf{H_{v^2}}\mathbf{H_{v^1}}\mathbf{A} = \mathbf{R} ,$$

where

$$\mathbf{A} = \begin{bmatrix} 3 & 20 & 1 \\ 4 & 20 & -1 \\ 0 & 3 & 2 \end{bmatrix} .$$

(a) Find the **unit** vector $\tilde{\mathbf{v}}^{\mathbf{1}} \in \mathbb{R}^3$ such that the first and second element of $\tilde{\mathbf{v}}^{\mathbf{1}}$ are both positive.

(b) Find the **unit** vector $\tilde{\mathbf{v}}^{\mathbf{2}} \in \mathbb{R}^3$ such that the second element of $\tilde{\mathbf{v}}^{\mathbf{2}}$ is negative and the third element is positive.

Solution:

$$\tilde{\mathbf{v}}^1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} , \quad \tilde{\mathbf{v}}^2 = \frac{1}{\sqrt{10}}\begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} .$$

[4 for each vector fully correct,0,0]

The reflecting vector $\mathbf{v}^1$ can be obtained as:

$$\mathbf{v^1} = \mathbf{a^1} + sign(\mathbf{a^1}_1)\|\mathbf{a^1}\|_2^2\, \mathbf{e^1}$$

$$= \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix} .$$

$$\tilde{\mathbf{v}}^1 = \frac{1}{\sqrt{80}}\begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix} .$$

The corresponding Householder matrix can be computed as:

$$\begin{aligned}\mathbf{H}_{\mathbf{v^1}} &= \mathbf{I}_3 - 2\frac{\mathbf{v^1}(\mathbf{v^1})^T}{(\mathbf{v^1})^T\mathbf{v^1}} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} \frac{4}{5} & \frac{2}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3}{5} & -\frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}.\end{aligned}$$

Premultiplying A by $\mathbf{H}_{\mathbf{v^1}}$ gives:

$$\mathbf{H}_{\mathbf{v^1}}\mathbf{A} = \begin{bmatrix} -5 & -28 & \frac{1}{5} \\ 0 & -4 & -\frac{7}{5} \\ 0 & 3 & 2 \end{bmatrix}.$$

Now we can obtain $\mathbf{v^2}$ as follows:

$$\mathbf{v^2} = \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} - 5\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -9 \\ 3 \end{bmatrix}.$$

$$\tilde{\mathbf{v}}^2 = \frac{1}{\sqrt{90}}\begin{bmatrix} 0 \\ -9 \\ 3 \end{bmatrix}.$$