Answer sheet compilation instructions

- Use only black or blue pen.

For open answers:

- Write clearly only **inside** boxes, away from the borders.
- Write a **single** character per box.
- Start writing from the left, leaving empty boxes on the right.

For multiple choice and true/false questions:

- **Fill** the circle for the answer you consider correct (only one answer is correct).
- Remarks and computations have **no** influence on points awarded.
- Any unclear or double marking will be considered as an answer not given (0 points).
- Wrong answers give **negative** points.

Exam instructions

- Turn off your devices and leave them in your bag.
- Only pens and Legi should be on the table.
- Fill last name and Legi number on the answer sheet.
- **Turn this sheet only when instructed to do so.**
- At the end of the exam, take everything except the single answer sheet which you want to submit.

Questions

NumCSE endterm, HS 2017

1. *Convolution* [5 P.]

   Let
   $$x = (1, 2, 3, 4, 5, 6, 4, 2, 0, 4, 8, 12).$$
   For an arbitrary vector y, let $x * y$ be the discrete convolution between x and y. Let $z[i]$ indicate the element at position i of any vector z, indexing starts from 0. For instance: $x[0] = 1$ and $x[3] = 4$.

   (a) If $y = (1, -1)$, what is $(x * y)[0]$?

   (b) If $y = (-1, 1)$, what is $(x * y)[7]$?

   (c) If $y = (-1, 2, -1)$, what is $(x * y)[12]$?

   (d) If $y = (1, 0, -1, 0, 1, 0, -1)$, what is $(x * y)[5]$?

   (e) If z is the discrete Fourier transform of x, what is $z[0]$?

   [1/0/0]

2. *Interpolation* [5 P.]

Consider the following code:

```
using namespace Eigen;

VectorXd f(const VectorXd &T, const VectorXd &Y) {
    int n = T.size();
    VectorXd tmp = VectorXd::Ones(n);
    MatrixXd V = MatrixXd::Zero(n, n);

    for (int j=0; j<n; j++) {
        V.col(j) = tmp;
        tmp = tmp.array()*T.array();
    }

    return V.fullPivLu().solve(Y);
}
```

(a) Choose the best asymptotic complexity of the function `f` for arbitrary inputs: [2/0/-1]

i) $O(\mathrm{n})$

ii) $O(\mathrm{n}^2)$

iii) $O(\mathrm{n}^3)$

iv) $O(\mathrm{n}^4)$

(b) (True/False) The function `f` returns a vector containing the coefficients of the polynomial obtained with Lagrange interpolation at nodes `T` with values `Y`. [1/0/-1]

(c) (True/False) Neglecting numerical instability, Lagrange and Newton interpolation are equivalent. [1/0/-1]

(d) (True/False) In case of numerical stability concerns, Lagrange interpolation is more robust than Newton interpolation. [1/0/-1]

3. *Bernstein polynomials* [7 P.]

   Let $B_j^n(t)$ denote the j-th Bernstein polynomial of order n, where n is an arbitrary positive integer.

   (a) Choose the correct definition of $B_j^n(t)$: [2/0/-1]

      (i) $\binom{n}{j} t^j (t-1)^{n-j}$

      (ii) $\binom{n}{j} t^j (1-t)^{n-j}$

      (iii) $\binom{j}{n} t^j (t-1)^{n-j}$

      (iv) $\binom{j}{n} t^j (1-t)^{n-j}$

   (b) (True/False) The Bernstein polynomials $B_0^n, \ldots, B_n^n$ constitute a basis of the space of polynomials of degree $n+1$. [1/0/-1]

   (c) (True/False) Exactly one polynomial among $B_0^n, \ldots, B_n^n$ attains value 1 in 0. [1/0/-1]

   (d) (True/False) It holds $\sum_{j=0}^{n} B_j^n(t) = 1$ for any $t \in \mathbb{R}$. [1/0/-1]

   (e) (True/False) It holds $B_j^n(t) \geq 0$ for any $t \in \mathbb{R}$. [1/0/-1]

   (f) (True/False) For any continuous function f, the Bernstein approximant of f converges exponentially in L^∞-norm to f. [1/0/-1]

4. *Function approximation* [9 P.]

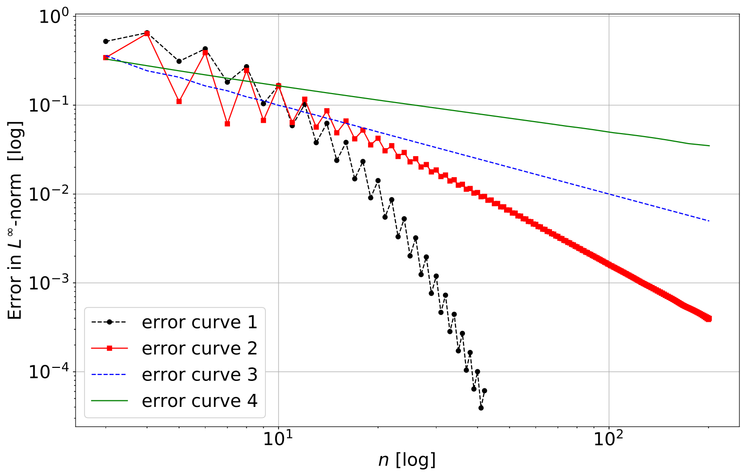
The following functions, defined for $x \in [-1, 1]$:

- $f_1(x) = 1/(1 + 16x^2)$
- $f_2(x) = \arcsin(x)$

are approximated using:

- $\mathcal{A}$: piecewise linear interpolation on an equidistant node set $\mathcal{J}_n = \{\frac{k}{n} : k = 0, 1, \ldots, n\}$, for $n \in \mathbb{N}$.
- $\mathcal{B}$: Chebychev interpolation, i.e. $f(x) \approx \mathcal{B}(f(x)) = \sum_{k=0}^{n} \alpha_k T_k(x)$, where $\alpha_k \in \mathbb{R}$ and T_k is the k-th Chebychev polynomial.

The error in L^∞-norm vs n plot is given below:



Assign the error curves to the corresponding approximation: [2/0/0]

		error curve 1	error curve 2	error curve 3	error curve 4
(a)	$\mathcal{A}(f_1)$	(i)	(ii)	(iii)	(iv)
(b)	$\mathcal{B}(f_1)$	(i)	(ii)	(iii)	(iv)
(c)	$\mathcal{A}(f_2)$	(i)	(ii)	(iii)	(iv)
(d)	$\mathcal{B}(f_2)$	(i)	(ii)	(iii)	(iv)

(e) For $f \in C^\infty([-1, 1])$, what type of convergence is expected for $\left\| f - \mathcal{A}(f) \right\|_{L^\infty([-1,1])}$ with respect to n? [1/0/0]

(i) algebraic (ii) exponential

5. *Quadrature formula* [6 P.]

(a) Consider the quadrature formula $Q(f) = \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f'(1/4)$ for the approximation of $I(f) = \int_0^1 f(x)dx$, where $f \in C^1([0, 1])$. If Q is of **order 3**, determine the coefficients α_k, for $k = 1, 2, 3$: [1/0/0]

(i) Value of $12\alpha_1$?

(ii) Value of $12\alpha_2$?

(iii) Value of $12\alpha_3$?

(b) Let

$$I_4(f) = \frac{1}{3}\{f(-1) + 2f(-1/2) + 2f(1/2) + f(1)\}$$

be a quadrature approximation of $I(f) = \int_{-1}^1 f(x)dx$, where $f \in C^0([-1, 1])$.

(i) What is the **order** of $I_4(f)$? [2/0/0]

(ii) (True / False) $I_4(f)$ is a Lagrange quadrature formula. [1/0/-1]