Answer sheet compilation instructions

- Use only black or blue pen.

For open answers:

- Write clearly only **inside** boxes, away from the borders.
- Write a **single** character (number or letter) per box.
- Start writing from the left, leaving empty boxes on the right.

For multiple choice and true/false questions:

- **Fill** the circle for the answer you consider correct (only one answer is correct).
- Remarks and computations have **no** influence on points awarded.
- Any unclear or double marking will be considered as an answer not given (0 points).
- Wrong answers give **negative** points.

Exam instructions

- Turn off your devices and leave them in your bag.
- Only pens and Legi should be on the table.
- Fill last name and Legi number on the answer sheet.
- **Turn this sheet only when instructed to do so.**
- At the end of the exam, take everything except the single answer sheet which you want to submit.
Questions
NumCSE endterm, HS 2017

1. Convolution [5 P]

Let

\[ x = (1, 2, 3, 4, 5, 6, 4, 2, 0, 4, 8, 12). \]

For an arbitrary vector \( y \), let \( x \ast y \) be the discrete convolution between \( x \) and \( y \). Let \( z[i] \) indicate the element at position \( i \) of any vector \( z \), indexing starts from 0. For instance: \( x[0] = 1 \) and \( x[3] = 4 \).

(a) If \( y = (1, -1) \), what is \( (x \ast y)[0] \)?

(b) If \( y = (-1, 1) \), what is \( (x \ast y)[7] \)?

(c) If \( y = (-1, 2, -1) \), what is \( (x \ast y)[12] \)?

(d) If \( y = (1, 0, -1, 0, 1, 0, -1) \), what is \( (x \ast y)[5] \)?

(e) If \( z \) is the discrete Fourier transform of \( x \), what is \( z[0] \)?

[1/0/0]
2. **Interpolation** [5 P.] 

Consider the following code:

```cpp
using namespace Eigen;

VectorXd f(const VectorXd &T, const VectorXd &Y) {
    int n = T.size();
    VectorXd tmp = VectorXd::Ones(n);
    MatrixXd V = MatrixXd::Zero(n, n);

    for (int j=0; j<n; j++) {
        V.col(j) = tmp;
        tmp = tmp.array()*T.array();
    }

    return V.fullPivLu().solve(Y);
}
```

(a) Choose the best asymptotic complexity of the function \( f \) for arbitrary inputs: [2/0/-1]

i) \( O(n) \)
ii) \( O(n^2) \)
iii) \( O(n^3) \)
iv) \( O(n^4) \)

(b) (True/False) The function \( f \) returns a vector containing the coefficients of the polynomial obtained with Lagrange interpolation at nodes \( T \) with values \( Y \). [1/0/-1]  

(c) (True/False) Neglecting numerical instability, Lagrange and Newton interpolation are equivalent. [1/0/-1]  

(d) (True/False) In case of numerical stability concerns, Lagrange interpolation is more robust than Newton interpolation. [1/0/-1]  

3. Bernstein polynomials [7 P.]

Let $B_n^j(t)$ denote the $j$-th Bernstein polynomial of order $n$.

(a) Choose the correct definition of $B_n^j(t)$: [2/0/-1]

(i) $\binom{n}{j} t^j (t-1)^{n-j}$

(ii) $\binom{n}{j} t^j (1-t)^{n-j}$

(iii) $\binom{n}{j} t^j (t-1)^{n-j}$

(iv) $\binom{n}{j} t^j (1-t)^{n-j}$

(b) (True/False) The Bernstein polynomials $B_n^0, \ldots, B_n^n$ constitute a basis of the space of polynomials of degree $n + 1$. [1/0/-1] ❌

(c) (True/False) For an arbitrary positive integer $n$, exactly one polynomial among $B_n^0, \ldots, B_n^n$ attains value 1 in 0. [1/0/-1] ✔

(d) (True/False) For an arbitrary positive integer $n$, $\sum_{j=0}^n B_j^n(t) = 1$ for any $t \in \mathbb{R}$. [1/0/-1] ✔

(e) (True/False) Fixed $n$, $B_n^j(t) \geq 0$ for any $t \in \mathbb{R}$. [1/0/-1] ❌

(f) (True/False) For any continuous function $f$, the Bernstein approximant of $f$ converges exponentially in $L^\infty$-norm to $f$. [1/0/-1] ❌
4. Function approximation

The following functions, defined for $x \in [-1, 1]$:
- $f_1(x) = 1/(1 + 16x^2)$
- $f_2(x) = \arcsin(x)$

are approximated using:
- $\mathcal{A}$: piecewise linear interpolation on an equidistant node set $\mathcal{J}_n = \{\frac{k}{n} : k = 0, 1, \ldots, n\}$, for $n \in \mathbb{N}$.
- $\mathcal{B}$: Chebychev interpolation, i.e. $f(x) \approx \mathcal{B}(f(x)) = \sum_{k=0}^{n} \alpha_k T_k(x)$, where $\alpha_k \in \mathbb{R}$ and $T_k$ is the $k$-th Chebychev polynomial.

The error in $L^\infty$-norm vs $n$ plot is given below:

![Error plot](image_url)

Assign the error curves to the corresponding approximation: [2/0/0]

<table>
<thead>
<tr>
<th></th>
<th>error curve 1</th>
<th>error curve 2</th>
<th>error curve 3</th>
<th>error curve 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>$\mathcal{A}(f_1)$</td>
<td>(i)</td>
<td>(ii)</td>
<td>(iii)</td>
</tr>
<tr>
<td>(b)</td>
<td>$\mathcal{B}(f_1)$</td>
<td>(i)</td>
<td>(ii)</td>
<td>(iii)</td>
</tr>
<tr>
<td>(c)</td>
<td>$\mathcal{A}(f_2)$</td>
<td>(i)</td>
<td>(ii)</td>
<td>(iii)</td>
</tr>
<tr>
<td>(d)</td>
<td>$\mathcal{B}(f_2)$</td>
<td>(i)</td>
<td>(ii)</td>
<td>(iii)</td>
</tr>
</tbody>
</table>

(e) For $f \in C^\infty([-1, 1])$, what type of convergence is expected for $\|f - \mathcal{A}(f)\|_{L^\infty([-1, 1])}$ with respect to $n$? [1/0/0]

- (i) algebraic
- (ii) exponential
5. Quadrature formula [6 P.]

(a) Consider the quadrature formula \( Q(f) = \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f'(1/4) \) for the approximation of \( I(f) = \int_0^1 f(x)dx \), where \( f \in C^1([0, 1]) \). If \( Q \) is of order 3, determine the coefficients \( \alpha_k \), for \( k = 1, 2, 3 \):

(i) Value of \( \alpha_1 \) ? \( 1 \)

(ii) Value of \( \alpha_2 \) ? \( 2 \)

(iii) Value of \( \alpha_3 \) ? \( 4 \)

(b) Let \( I_4(f) = \frac{1}{3} \{ f(-1) + 2f(-1/2) + 2f(1/2) + f(1) \} \) be a quadrature approximation of \( I(f) = \int_{-1}^1 f(x)dx \), where \( f \in C^0([-1, 1]) \).

(i) What is the order of \( I_4(f) \)? \( 2 \)

(ii) (True / False) \( I_4(f) \) is a Lagrange quadrature formula. \( \text{False} \)