Referring to the cubic spline computation:

We have 2 pieces of information from the interpolating conditions:

1. \[ s_d(t_{j-1}) = \tilde{\theta}_d = \gamma_{d-1} \]
   \[ \rightarrow \text{this determines } \tilde{\gamma}_d \]

2. We also have \[ s_d(t_j) = \gamma_d \]

   and also \[ s_d(t_j) = \tilde{\gamma}_d + \tilde{b}_d (t_j - t_{j-1}) + \tilde{c}_d (t_j - t_{j-1})^2 \]
   \[ + \tilde{d}_d (t_j - t_{j-1})^3 \]

   \[ \gamma_d \]

We already have checked \[ \tilde{\gamma}_d = \frac{\theta_d - \theta_{d-1}}{2} \]
\[ \tilde{b}_d = \frac{\theta_d - \theta_{d-1}}{2 \gamma_j} \]
and
\[ \tilde{\gamma}_d = \gamma_{d-1} \]

Plug this into (*):

\[ \gamma_{d-1} + \tilde{b}_d \gamma_j + \frac{\theta_d - \theta_{d-1}}{2} \gamma_j^2 + \frac{\theta_d - \theta_{d-1}}{6 \gamma_j} \gamma_j^3 = \gamma_d \]
\[ \beta_d h_d = Y_d - Y_{d-1} - h_d^2 \left( \frac{2 \sigma_{d-1}^2 + \sigma_d^2}{6} \right) \]

\[ \tilde{b}_d = \frac{Y_d - Y_{d-1}}{h_d} - h_d \left( \frac{2 \sigma_{d-1}^2 + \sigma_d^2}{6} \right) \]

Equation for \( \tilde{b}_d \) verified.

Next: We use the matching of the 1st derivatives:

\[ \tilde{b}_d + 2 \tilde{c}_d^2 h_d + 3 \tilde{d}_d h_d^2 = \tilde{b}_{d+1} \]

to formulate an LSE for the \( \sigma_d^2 \)s.

We write

\[ \frac{Y_d - Y_{d-1}}{h_d} - \frac{h_d (2 \sigma_{d-1}^2 + \sigma_d^2)}{6} + \frac{\sigma_{d-1}^2 h_d^2}{2} + \frac{\sigma_d^2 - \sigma_{d-1}^2}{2} h_d^2 \]

\[ = \frac{Y_{d+1} - Y_d}{h_{d+1}} - \frac{h_{d+1} (2 \sigma_d^2 + \sigma_{d+1}^2)}{6} \]
\[
\begin{align*}
\Rightarrow & \quad c_{i-j-1} \left( -\frac{h_i}{3} + h_j - \frac{h_i}{2} \right) + c_{i-j} \left( -\frac{h_i}{6} + \frac{h_j}{2} + \frac{h_{i+j}}{3} \right) \\
& \quad + c_{i+j} \frac{h_{i+j}}{6} = \frac{Y_{i+j} - Y_{i-j}}{h_{i+j}} - \frac{Y_i - Y_{i-1}}{h_{i+j}} \quad = \hat{\beta}_i
\end{align*}
\]

Then we obtain the LSE from the lecture.