Numerical Methods for
Computational Science and Engineering

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We want \( QF \) on \([a, b]\):

Approximate
\[
\int_a^b f(t) \, dt
\]

\[\downarrow\]

\[
\int_a^b f(t) \, dt = \int_{-1}^1 f(\varphi(t)) \varphi'(t) \, dt
\]

Can be transformed to \([-1, 1]\):

Suppose we know a \( QF \): \((\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_N)\) on \([-1, 1]\)

\[
QF(f) = \frac{2}{N-1} \sum_{i=1}^{N-1} f(\hat{\xi}_i)
\]

\[\varphi(t) = \frac{1}{2} (1-t) a + \frac{1}{2} (1+t) b\]
\[ \int_{a}^{b} f(t) \, dt \approx \frac{1}{2} (b-a) \sum_{j=1}^{n} \omega_j \, f(c_j) \]

\[ = \frac{1}{2} \sum_{j=1}^{n} \omega_j \, f(c_j) \quad (\hat{f} = f \circ \Phi) \]

\[ c_j = \frac{1}{2} \left( 1 - \hat{c}_j \right) a + \frac{1}{2} \left( 1 + \hat{c}_j \right) b \]

\[ \omega_j = \frac{1}{2} (b-a) \hat{\omega}_j \]

\((c_j, \omega_j)\) QF for \([a, b]\)

\((\hat{c}_j, \hat{\omega}_j)\) QF for \([-1, 1]\)

Quadrature by approximation schemes

interpolation schemes \rightarrow approximation schemes \rightarrow quadrature schemes

Suppose we have an approximation scheme

\[ A : C^0([a, b]) \rightarrow V \]

space of "simple" functions

Then, we want to perform numerical integration as:

\[ \int_{a}^{b} f(t) \, dt \approx \int_{a}^{b} (A \hat{f})(t) \, dt =: Q_A(f) \]
Use interpolation schemes:

\[ I_f(f) \quad T = \{ t_1, \ldots, t_n \} \] node set

\[ \int_a^b f(t) \, dt \quad \text{instead:} \]

\[ \approx \int_a^b \left[ f(t_1), \ldots, f(t_n) \right]^\top (t) \, dt \]

\[ = \sum_{i=1}^n f(t_i) \omega_i \]

For this to work, the interpolation operator needs of \( I_f \) to be linear.

\[ \int_a^b \left[ f(t_1), \ldots, f(t_n) \right]^\top (t) \, dt \]

\[ = \sum_{i=1}^n \int_a^b f(t_i) \, dt \omega_i \]

\[ = \sum_{i=1}^n f(t_i) \omega_i \]
Quality of the interpolation gives a bound on the error of the QF:

\[ E_n(f) = \left| \int_a^b (f(t) - \sum_{i=0}^{n-1} f(t_i) \mathcal{L}_i(t)) \, dt \right| \]

\[ \leq |b-a| \cdot \| f - \sum_{i=0}^{n-1} f(t_i) \mathcal{L}_i(t) \|_{L^\infty[a,b]} \]

**Interpolation error**

### Polynomial QFs

**Idea:**

\[ \int_a^b f(t) \, dt \approx Q_n(f) := \int_a^b p_{n-1}(t) \, dt \]

\[ p_{n-1} \text{ polynomial Lagrange interpolant of } f \text{ for a given node set } \]

\[ \mathcal{J} := \{ t_0, \ldots, t_{n-1} \} \subset [a, b] \]

**Recall Lagrange polynomial:**

\[ \mathcal{L}_i(t) = \prod_{j=0}^{n-1} \frac{t - t_j}{t_i - t_j}, \quad \text{for } j \neq i \]

\[ p_{n-1}(t) = \sum_{i=0}^{n-1} f(t_i) \mathcal{L}_i(t) \]
QF: \[ \int_{a}^{b} p_{n-2}(t) \, dt = \int_{a}^{b} \left( \sum_{i=0}^{n-1} f(t_i) \, L_i(t) \right) \, dt \]

\[ = \sum_{i=0}^{n-1} f(t_i) \int_{a}^{b} L_i(t) \, dt \]

\[ = \sum_{i=1}^{n} f(t_{i-1}) \int_{a}^{b} L_i(t) \, dt \]

Weights: \[ \omega_i := \int_{a}^{b} L_i(t) \, dt \]

\[ c_i := t_{i-1} \]

Examples:

- \( n = 1 \) midpoint rule

Approximation of \( f \) by a constant polynomial:

\[ Q_m(f) = (b-a) \cdot f(t_0) \]

\[ t_0 = \frac{1}{2} (a+b) \]

More generally: Newton–Cotes formulas

\( n \)-point Newton–Cotes formula
Equidistant nodes & Lagrange interpolation

\[ b_i = a + \frac{b-a}{n-1} i \quad i = 0, \ldots, n-1 \]

\[ n = 2 \] trapezoidal rule

\[ \omega_i = \int_a^b L_i(t) \, dt \]

\[ \omega_1 = \int_a^b L_0(t) \, dt = \int_a^b \frac{t-a}{b-a} \, dt = \frac{b-a}{2} \]

\[ \omega_2 = \int_a^b L_1(t) \, dt = \int_a^b \frac{t-b}{b-a} \, dt = \frac{b-a}{2} \]

Approximation of \( f \) by a linear polynomial

\[ Q_1(f) = \frac{b-a}{2} (f(a) + f(b)) \quad n = 3 \] Simpson rule

\[ = \omega_1 f(a) + \omega_2 f(b) \]

\[ \int_a^b f(t) \, dt \approx \frac{b-a}{6} \left( f(a) + 4 f\left( \frac{a+b}{2} \right) + f(b) \right) \]
Polynomial QF based on Lagrange interpolation is straightforward. **BUT:** Lagrange interpolation with equidistant nodes is numerically unstable.

**Remedy:** Use Chebyshev nodes instead → yield Clenshaw-Curtis QF.

> **Gauss Quadrature**

This requires a concept of quality for QFs.

→ **Order of a QF**

**Definition 6.3.1 (Order of a quadrature rule).** The order of quadrature rule \( Q_n : C^0([a, b]) \to \mathbb{R} \) is defined as

\[
\text{order}(Q_n) = \max\{m \in \mathbb{N}_0 : Q_n(p) = \int_a^b p(t) \, dt \quad \forall p \in P_m\} + 1,
\]

(6.11)

that is, as the maximal degree +1 of polynomials for which the quadrature rule is guaranteed to be exact.

\[
\int_a^b f(t) \, dt \approx \sum_{i=1}^{n} w_i f(x_i)
\]

up to which degree is this exact for polynomials.
Note: This concept of order of a QF
is invariant under affine transformations
(such as \( \delta \) from before to transform
from \([-1,1]\) to \([a,b]\)).

Example: Suppose we have a polynomial QF
with \( n \) points.
What can we say about its order?

Polynomial QF with \( n \) points:
is based on polynomial interpolation with
\( n \) nodes.

So: any \( p \in \mathbb{P}_{n-1} \) is uniquely determined by
\( n \) values \( p(t_1), \ldots, p(t_n) \)

\( \Rightarrow \) interpolation gives \( p \).
\( \Rightarrow \) quadrature is exact for any \( p \in \mathbb{P}_{n-1} \)

This means \( \text{order} (QF_{py}) \geq (n-1)+1 = n \)

[In fact, we will see that \( > n \) is possible]
Characterization of $n$-point QFs with order $\geq n$:

**Theorem 6.3.1** (Sufficient order conditions for quadrature rules). An $n$-point quadrature rule on $[a,b]$ (see Definition 6.1.1)

$$Q_n(f) := \sum_{j=1}^{n} w_j f(t_j), \quad f \in C^0([a,b]),$$

with nodes $t_j \in [a,b]$ and weights $w_j \in \mathbb{R}$, $j = 1, \ldots, n$, has order $\geq n$, if and only if

$$w_j = \int_{t_{j-1}}^{b} L_{j-1}(t) \, dt, \quad j = 1, \ldots, n,$$

where $L_k$, $k = 0, \ldots, n-1$, is the $k$-th Lagrange polynomial (5.14) associated with the ordered node set $\{t_1, t_2, \ldots, t_n\}$.

Therefore: For QF $Q_n$ to have order $\geq n$ weights $w_j$ only depend on node set

$$T = \{t_1, \ldots, t_n\}$$

**Proof of Theorem 6.3.1**: Exercise

The big question: Are there $n$-point QFs with order $\geq n$?

Upper bound to the order of an $n$-point QF:

**Theorem 6.3.2** (Maximal order of $n$-point quadrature rule). The maximal order of an $n$-point quadrature rule is $2n$.

Why? To see this, we construct a polynomial

$$q \in \mathbb{P}_{2n} \quad \text{s.t.} \quad Q_n(q) = \int_{a}^{b} q(t) \, dt.$$
Because then: order $< 2n + 1$

or $\leq 2n$. 

How?

$$Q_n(f) := \sum_{i=1}^{n} w_i f(x_i^n)$$

with choice

$$q(t) := (t-c_1^n)^2(t-c_2^n)^2 \ldots (t-c_n^n)^2 \in \mathbb{P}_{2n}$$

$$= t^{2n} + \ldots$$

$$q(t) \geq 0 \quad \text{(by definition)}$$

$$\Rightarrow \int_{a}^{b} q(t) \, dt > 0$$

On the other hand:

$$Q_n(g) = \sum_{i=1}^{n} w_i g(x_i^n) = 0$$

$$\Rightarrow Q_n(g) = \int_{a}^{b} g(t) \, dt$$

$$\Rightarrow \text{order}(Q_n) < 2n + 1$$

$$\Rightarrow \text{order}(Q_n) \leq 2n$$

Example: 2-point QF $Q_2$ of order 4 $\mathbb{P}_{[-1,1]}$

$$Q_2(p) = \int_{a}^{b} p(t) \, dt \quad \forall p \in \mathbb{P}_2$$

$$\Leftrightarrow Q_2\left(\left\{ t \mapsto t^{q_i} \right\} \right) = \frac{1}{q+1} \left( b^{q+1} - a^{q+1} \right)$$

$q = 0, 1, 2, 3$
This is sufficient because monomials \((t_0^{t_9})^n\) form a basis of \(B_n\).

Simple check: \(c_1, c_2, \omega_1, \omega_2\) are all \(\neq 0\)

\(a = -1, \quad b = 1:\)

\[ QF: \quad \int_{-1}^{1} f(t) \, dt = \omega_1 f(c_1) + \omega_2 f(c_2) \]

\[ \Rightarrow \int_{-1}^{1} 1 \, dt = 2 = \omega_1 + \omega_2 \quad (1) \]

\[ \Rightarrow \frac{c_2}{c_1} = \frac{c_3^3}{c_4^3} \quad \Rightarrow \quad \frac{c_2^2}{c_1^2} = 1 \]

\[ \Rightarrow \frac{c_4^2}{c_2^2} = 1 \]

Use this in (3):

\[ \int_{-1}^{1} t \, dt = \frac{t^2}{2} \bigg|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0 = \omega_1 c_1 + \omega_2 c_2 \quad (2) \]

\[ \Rightarrow \frac{2}{3} = c_1^2 \frac{(\omega_1 + \omega_2)}{} \]

\[ \Rightarrow \frac{2}{3} = c_1^2 \frac{(\omega_1 + \omega_2)}{2} \]

\[ \Rightarrow c_1^2 = \frac{1}{3} \quad c_2^2 = \frac{4}{3} \]

\[ \int_{-1}^{1} t^2 \, dt = \frac{t^3}{3} \bigg|_{-1}^{1} = \frac{2}{3} = \omega_1 c_1^2 + \omega_2 c_2^2 \quad (3) \]

\[ \Rightarrow c_1^2 = \frac{1}{3} \quad c_2^2 = \frac{4}{3} \]

\[ \int_{-1}^{1} t^3 \, dt = 0 = \omega_1 c_1^3 + \omega_2 c_2^3 \quad (4) \]
Choose \( c_1, c_2 \) s.t. \( c_1 + c_2 \Rightarrow c_1 = -c_2 \)

\[ (2) : \quad \omega_1 c_1 + \omega_2 c_2 = 0 \]
\[ \omega_1 = \omega_2 = 0 \]
\[ \Rightarrow \quad \omega_1 = \omega_2 \]

Pick \( c_1 = -\frac{1}{13}, \quad c_2 = \frac{1}{13} \)

\[ (1) : \quad \omega_1 + \omega_2 = 2 \]
\[ \Rightarrow \quad \omega_1 = 1, \quad \omega_2 = 1 \]

\( \text{Q}_2: 2\text{-point QF of order 4:} \)

\[ \text{Q}_2(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \]

More generally we can ask:

For any \( n \in \mathbb{N} \), is there \( \text{Q}_n \) s.t. \( \text{Q}_n \) is \( n\)-point and of order \( 2n \)?

**Theorem 6.3.3** (Existence of \( n\)-point quadrature formulas of order \( 2n \)). Let \( \{\text{Q}_n\}_{n \in \mathbb{N}_0} \) be a family of non-zero polynomials that satisfies

- \( \text{Q}_n \in \mathbb{P}_n \),
- \( \int_{-1}^{1} q(t) \text{P}_n(t) \, dt = 0 \) for all \( q \in \mathbb{P}_{n-1} \) (\( L^2([-1,1]) \)-orthogonality),
- The set \( \{c_j^{(n)}\}_{j=1}^{m} \), \( m \leq n \), of real zeros of \( \text{P}_n \) is contained in \([-1,1]\).

Then the quadrature rule (see Definition 6.1.1)

\[ \text{Q}_n(f) := \sum_{j=1}^{m} w_j f(c_j^{(n)}) \]

with weights chosen according to **Theorem 6.3.1** provides a quadrature formula of order \( 2n \) on the interval \([-1,1]\).

We will see: \( m = n \)

\[ \text{P}_n(t) = (t-c_1^{(n)}) \cdots (t-c_n^{(n)}) \]

\( n\)-point QF of order \( 2n \)
It turns out: Up to scaling factors, the $P_n$'s are the Legendre polynomials.

**Definition 6.3.2 (Legendre polynomials).** The $n$-th Legendre polynomial $P_n$ is defined by:

- $P_n \in \mathbb{P}_n$,
- $\int_{-1}^{1} P_n(t)q(t) \, dt = 0 \quad \forall q \in \mathbb{P}_{n-1}$,
- $P_n(1) = 1$.

**Fact:** $P_n$ has $n$ distinct zeros in $(-1,1)$.

**Why does $P_n$ have $n$ distinct zeros in $(-1,1)$?**

What happens if $P_n$ has only $m < n$ zeros?

\[ \xi_1, \ldots, \xi_m \in (-1,1) \]
Observe: $P_n$ changes sign at $x_1, \ldots, x_m$.

Now define $g(t) = \prod_{i=1}^m (t - x_i) \in P_m \subseteq P_{n-2}$ and we know by definition of $P_n$:

$$\int_{-1}^1 P_n(t)g(t)\,dt = 0$$

$$\langle P_n, g \rangle_{L^2([-1,1])}$$

But: $g$ also changes sign at $x_1, \ldots, x_m$!

$P_n(t) \cdot g(t)$ cannot change sign on $(-1,1)$

$\Rightarrow P_n \cdot g > 0$ on $(-1,1)$ or $P_n \cdot g < 0$ on $(-1,1)$

$$\Rightarrow \int_{-1}^1 P_n(t)g(t)\,dt < 0$$

Definition 6.3.3 (Gauss-Legendre quadrature formulas). The $n$-point Quadrature formulas whose nodes are given by the zeros of the $n$-th Legendre polynomial (see Definition 6.3.2), and whose weights are chosen according to Theorem 6.3.1, are called Gauss-Legendre quadrature formulas.

Note: The weights of the G-L QFs are positive.
Derivation of positivity:

We denote the Gauss points by \( x_j^n \), \( j = 1, \ldots, n \) for an \( n \)-point \( G-L \) QF.

Define \( q_k(t) := \frac{1}{n} \prod_{j=1,j \neq k}^{n} (t - x_j^n)^2 \)

\[ \Rightarrow q_k \in P_{2n-2} \Rightarrow \text{\( n \)-point \( G-L \) QF} \]

integrates \( q_k \) exactly!

\[ 0 < \int_{-1}^{1} q_k(t) \, dt = \sum_{j=1}^{n} \omega_j^n \cdot q_k(x_j^n) = \omega_k^n \cdot q_k(x_k^n) \]

\[ \quad \frac{\text{QF}}{\text{QF}} > 0 \]

\[ \Rightarrow \omega_k^n > 0. \quad \text{This is true for all} \quad k = 1, \ldots, n \]

As for Chebyshev polynomials, we again have a 3-term recursion formula:

Recursive formula for Legendre polynomials

Legendre polynomials satisfy the 3-term recursion (similar to Chebyshev polynomials).

\[ P_{n+1}(t) := \frac{2n+1}{n+1} t P_n(t) - \frac{n}{n+1} P_{n-1}(t), \quad P_0 := 1, \quad P_1(t) := t. \quad (6.14) \]
Simple bound on the quadrature error:

(Using positivity of weights)

**Theorem 6.3.4 (Quadrature error estimate for quadrature rules with positive weights).** For every \( n \)-point quadrature rule \( Q_n \) as in (6.1) of order \( q \in \mathbb{N} \) with weights \( w_j \geq 0, j = 1, \ldots, n \) the quadrature error satisfies

\[
E_n(f) := \left| \int_a^b f(t) \, dt - Q_n(f) \right| \leq 2|b-a| \inf_{p \in P_{q+1}} \| f - p \|_{L^\infty([a,b])} \quad \forall f \in C^q([a,b]).
\]  

(6.15)  

**Lemma 6.3.3 (Quadrature error estimates for \( C^r \)-integrands).** For every \( n \)-point quadrature rule \( Q_n \) as in (6.1) of order \( q \in \mathbb{N} \) with weights \( w_j \geq 0, j = 1, \ldots, n \) we find that the quadrature error \( E_n(f) \) for an integrand \( f \in C^q([a,b]), q \in \mathbb{N}_0 \), satisfies

in the case \( q \geq r \):

\[
E_n(f) \leq C q^{-q} |b-a|^{r+1} \left\| \frac{f^{(r)}}{r!} \right\|_{L^\infty([a,b])},
\]

(6.16)  

in the case \( q < r \):

\[
E_n(f) \leq \frac{|b-a|^{r+1}}{1!} \left\| \frac{f^{(r)}}{r!} \right\|_{L^\infty([a,b])},
\]

(6.17)  

with a constant \( C > 0 \) independent of \( n, f \), and \([a,b] \).

\[
\text{(Key ingredient: } \sum_{j=1}^n w_j \mid \sum_{j=1}^n w_j = 1 \text{)}
\]

**Question:** Asymptotic behavior of \( E_n(f) \) as \( n \to \infty \)

\[
\text{quadrature error}
\]

\[
f \in C^r([a,b]) : E_n(f) = \Theta(n^{-r}) \quad \text{alg. conv.}
\]

If \( f \in C^\infty([a,b]) : E_n(f) = \Theta(\lambda^n) \quad \text{exp. conv.}
\]

\[\lambda \in (0,1)\]
Example: 2 functions

- smooth
- only cont.

**Numerical quadrature of function $f_1(t) := \frac{1}{1 + (5t)^2}$ on $[0, 1]$**

**Numerical quadrature of function $f_2(t) := \sqrt{t}$ on $[0, 1]$**

**Remark:**

The integral of this particular function can be transformed.

Substitute $s = \sqrt{t}$
\[ \text{What do the asymptotics actually tell us?} \]

Suppose for fixed \( f \) we have sharp asymptotics

\[ E_n(f) = \Theta(n^{-r}) \Rightarrow E_n(f) \approx C \cdot n^{-r} \]

Now: We want to decrease the quadrature error by factor \( g > 1 \)

Then:

\[ \frac{C \cdot n_\text{old}^{-r}}{C \cdot n_\text{new}^{-r}} = g \Rightarrow n_\text{new} = n_\text{old} \cdot g^{1/r} \]

This tells how many more points we need to take to improve quadrature error by a factor \( g \).
Suppose we had \( E_n(f) \approx C \cdot 1^n \) instead of \( E_n(f) \approx C \cdot 1^n \).

\[
C \cdot 1 \text{ old} = \int \\
C \cdot 1 \text{ new} = \int \\

V_{\text{new}} = V_{\text{old}} + \left[ \frac{\log S}{\log \lambda} \right]
\]

Here: only need to add a fixed number of nodes to improve the quadrature error by a factor \( g \).

**Composite Quadrature**

As for interpolation: introduce a mesh

2. then apply a QF on each cell

Mesh: \( M = \{ a = x_0 < x_1 < \ldots < x_m = b \} \)

\[
\int_a^b f(t) \, dt = \sum_{j=1}^{m} \int_{x_{j-1}}^{x_j} f(t) \, dt
\]

on each interval \( I_j := [x_{j-1}, x_j] \) apply an \( n_j \)-point QF

total number of function evaluations \( f : \sum_{j=1}^{n} n_j \).
Motivation:  • nodes are not freely choosable
               (then: equidistant + global polygon interp
                is a bad idea)
               • function is e.g. highly oscillatory

Examples:

Composite trapezoidal rule, cf. (7.2.5)

$$\int_a^b f(t)dt = \frac{1}{2}(x_1 - x_0)f(a) + \sum_{j=1}^{m-1} \frac{1}{2}(x_{j+1} - x_{j-1})f(x_j) + \frac{1}{2}(x_m - x_{m-1})f(b).$$ \hfill (7.4.4)

Composite Simpson rule, cf. (7.2.6)

$$\int_a^b f(t)dt = \frac{1}{3}(x_1 - x_0)f(a) + \sum_{j=1}^{m-1} \frac{1}{3}(x_{j+1} - x_{j-1})f(x_j) + \sum_{j=1}^{m} \frac{2}{3}(x_j - x_{j-1})f(\frac{1}{2}(x_j + x_{j-1})) + \frac{2}{3}(x_m - x_{m-1})f(b).$$ \hfill (7.4.5)

Error estimates for composite QP:

\[ \rightarrow \text{add errors on each } I_j.\]

Suppose on each $I_j$: QP $Q^*_p$ of order $q$ has positive weights
If \( f \in C^r ([x_{i-1}, x_i]) \):

\[
\left| \int_{x_{i-1}}^{x_i} f(t) \, dt - Q_n^f (x_i, x_{i-1}) \right| \leq C \cdot h_j^{\min\{r, q\} + 1} \cdot \| f(\min\{r, q\}) \|_{L^\infty(I_j)}
\]

\[\text{(6.17)}\]

\[\Rightarrow\]

\[
\lesssim \sum_{j=1}^{m} \left| \int_{x_{i-1}}^{x_i} f(t) \, dt - Q_n^f (t) \right| \leq \sum_{j=1}^{m} \left| \int_{x_{i-1}}^{x_i} f(t) \, dt - Q_n^f (x) \right|
\]

\[\leq C \cdot \sum_{j=1}^{m} h_j^{\min\{r, q\} + 1} \| f(\min\{r, q\}) \|_{L^\infty(I_j)}
\]

\[\leq C \cdot h_{\mu}^{\min\{r, q\}} \max_{j=1, \ldots, m} \| f(\min\{r, q\}) \|_{L^\infty(I_j)} \cdot \sum_{j=1}^{m} h_j
\]

\[\text{where} \quad h_{\mu} = \max_{r=1, \ldots, m} h_r
\]

\[\Rightarrow\] algebraic convergence in mesh width \( h_j \)

\[\text{"} h^{-\text{convergence}} \text{"}
\]

For \( r > q \): algebraic convergence in \( h_j \) of rate \( q \) (=order of the QP)

Example: Composite trapezoidal: \( q = 2 \)
Simpson: \( q = 4 \)
(higher than expected)

For sufficiently many times differentiable functions (i.e. \( r > q \)): \( O(h^2) \) vs. \( O(h^4) \)

trap. Simpson
quadrature error, $f_1(t) := \frac{1}{1 + (5t)^2}$ on $[0, 1]$

$\log - \log - \text{plot}$

quadrature error, $f_2(t) := \sqrt{t}$ on $[0, 1]$

$\in \mathcal{C}^0([0, 1])$

$O(h^{3/2})$
Comparison of asymptotic rates of

\[ f \in C^r([a,b]) : \text{Composite QF: } \Theta(n^{-\min(r,q)}) \]
\[ \text{Gauss QF: } \Theta(n^{-r}) \]

\[ \Rightarrow \text{Gauss is at least as good as composite QF} \]
\[ \theta \text{ achieves best possible rate} \]

\[ f \in C^\infty([a,b]) : \text{Composite QF: } \Theta(n^{-q}) \text{ alg.conv.} \]
\[ \text{Gauss QF: } \Theta(1^n) \text{ exp.conv.} \]
\[ \lambda \in \mathbb{Q}(a,b) \]