Numerical Methods for Computational Science and Engineering

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Numerical integration on arbitrary intervals $[a,b]$

$\rightsquigarrow$ we can reduce this to finding QF on a reference interval, e.g. $[-1,1]$

Suppose: We know a QF: $(\hat{c}_j, \hat{\omega}_j)_{j=1}^{n}$ on $[-1,1]$

$$\left[QF(f) = \sum_{j=1}^{n} \hat{\omega}_j f(\hat{c}_j) \right]$$

We want QF on $[a,b]$:

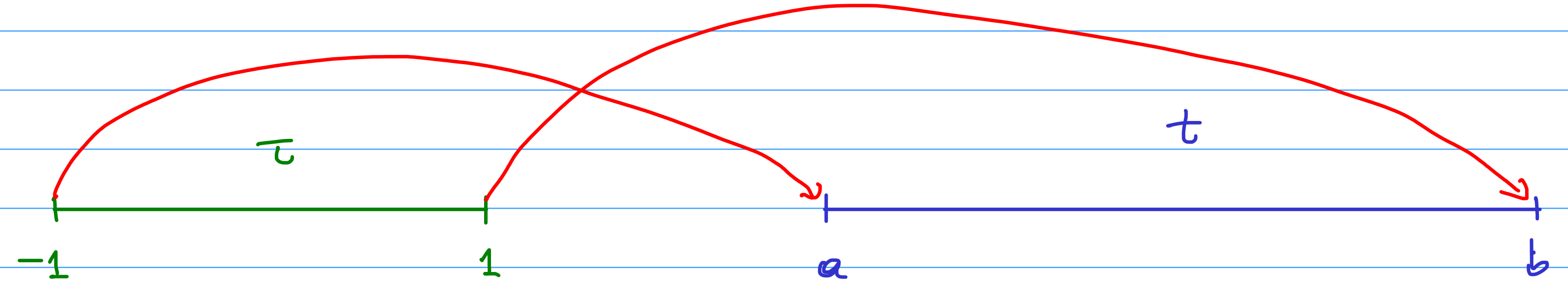
Approximate $\int_a^b f(t)\,dt$!

$\downarrow$

can be transformed to $[-1,1]$:

$$\int_a^b f(t)\,dt = \int_{-1}^{1} f(\Phi(\tau))\,\Phi'(\tau)\,d\tau$$

$$\Phi(\tau) = \frac{1}{2}(1-\tau)\,a + \frac{1}{2}(1+\tau)\,b$$



$$\int_a^b f(t)\,dt \approx \overbrace{\frac{1}{2}(b-a)}^{\Phi'(\tau)} \sum_{j=1}^{n} \hat{\omega}_j \hat{f}(\hat{c}_j) \qquad (\hat{f} = f \circ \Phi)$$

$$= \sum_{j=1}^{n} \omega_j f(c_j)$$

$$c_j = \frac{1}{2}(1-\hat{c}_j)\,a + \frac{1}{2}(1+\hat{c}_j)\,b$$

$$\omega_j = \frac{1}{2}(b-a)\,\hat{\omega}_j$$

(c_j, ω_j) QF for $[a,b]$

$(\hat{c}_j, \hat{\omega}_j)$ QF for $[-1,1]$

Quadrature by approximation schemes

interpolation schemes	$\longrightarrow$	approximation schemes	$\longrightarrow$	quadrature schemes

Suppose we have an approximation scheme

$$A : C^0([a,b]) \longrightarrow V$$

space of "simple" functions

Then, we want to perform numerical integration as:

$$\int_a^b f(t)\,dt \approx \int_a^b (Af)(t)\,dt =: Q_A(f)$$

Use interpolation schemes:

$I_{\mathcal{T}}(f)$ $\qquad \mathcal{T} = \{t_1, \dots, t_n\}$ node set

$\uparrow$ interpolation of $f(t_1), \dots, f(t_n)$

$$\int_a^b f(t)\,dt \quad \text{instead:}$$

$$\approx \int_a^b I_{\mathcal{T}}\left[f(t_1), \dots, f(t_n)\right]^T (t)\,dt$$

For this to work, the interpolation operator needs to be linear.

$$\int_a^b I_{\mathcal{T}}\left[f(t_1), \dots, f(t_n)\right]^T (t)\,dt$$

$$= \int_a^b I_{\mathcal{T}}\left[\sum_{i=1}^n f(t_i) e_i\right]^T (t)\,dt$$

$$\uparrow \quad \left[f(t_1), \dots, f(t_n)\right] = \sum_{i=1}^n f(t_i) e_i$$

$$= \sum_{i=1}^n f(t_i) \underbrace{\int_a^b I_{\mathcal{T}}[e_i](t)\,dt}_{=: \omega_i} = \sum_{i=1}^n f(t_i)\,\omega_i$$

linearity of $I_{\mathcal{T}}$

this is a QF

Quality of the interpolation gives a bound on the error of the QF

$$E_n(f) = \left| \int_a^b \left(f(t) - I_{\mathcal{T}}[f(t_1),\dots,f(t_n)]^T(t) \right) dt \right|$$

$$\leq |b-a| \cdot \underbrace{\| f - I_{\mathcal{T}}[f(t_1),\dots,f(t_n)]^T \|_{L^\infty([a,b])}}_{\text{interpolation error}}$$

Polynomial QFs

Idea:

$$\int_a^b f(t)\,dt \approx Q_n(f) := \int_a^b p_{n-1}(t)\,dt$$

p_{n-1} polynomial Lagrange interpolant of f for a given node set $\mathcal{T} := \{t_0,\dots,t_{n-1}\} \subset [a,b]$

$$p_{n-1}(t) = \sum_{i=0}^{n-1} f(t_i) \cdot L_i(t)$$

Recall Lagrange polyn. $L_i(t) = \prod_{\substack{j=0 \\ j \neq i}}^{n-1} \frac{t-t_j}{t_i-t_j}$

QF: $\int_a^b p_{n-1}(t)\,dt = \int_a^b \left(\sum_{i=0}^{n-1} f(t_i) L_i(t)\right) dt$

$= \sum_{i=0}^{n-1} f(t_i) \cdot \int_a^b L_i(t)\,dt$

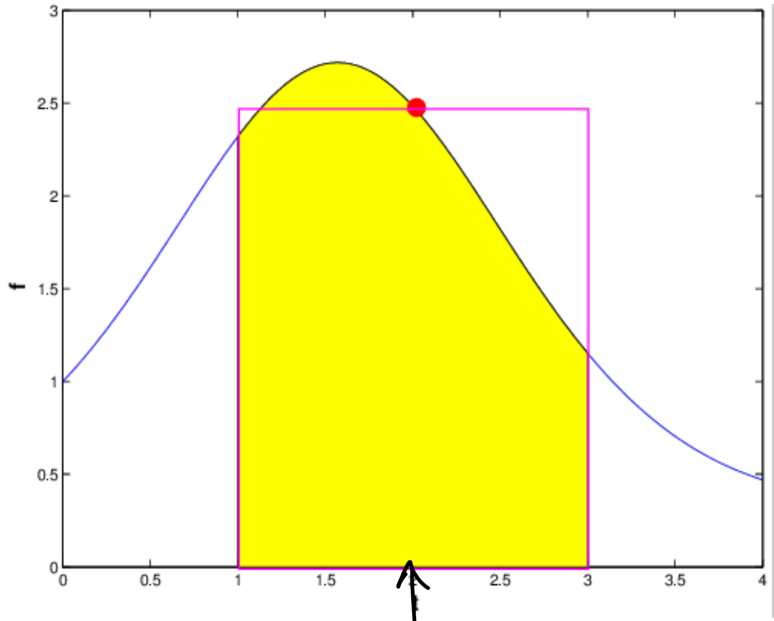
$= \sum_{i=1}^{n} f(t_{i-1}) \int_a^b L_{i-1}(t)\,dt$

weights: $\omega_i := \int_a^b L_{i-1}(t)\,dt$

$c_i := t_{i-1}$

Examples:

$n = 1$ midpoint rule



Approximation of f by a constant polynomial

$Q_{mp}(f) = (b-a) \cdot f(t_0)$

$t_0 = \frac{1}{2}(a+b)$

More generally: Newton-Cotes formulas

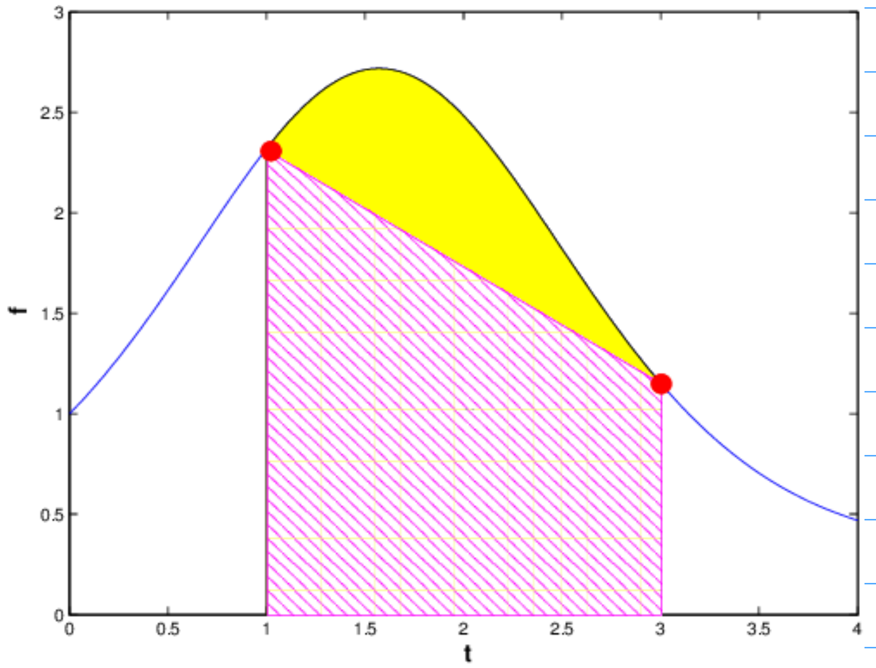
n-point Newton-Cotes formula

Equidistant nodes & Lagrange interpolation

$\downarrow$

$$t_j := a + \frac{b-a}{n-1} j \qquad j = 0, \dots, n-1$$

$n = 2$ trapezoidal rule



Approximation of by by a linear polynomial

$$Q_{tr}(f) = \frac{b-a}{2}\left(f(a) + f(b)\right)$$

$$= \omega_1 f(a) + \omega_2 f(b)$$

$$\omega_i = \int_a^b L_{i-1}(t)\,dt$$

$$\omega_1 = \int_a^b L_0(t)\,dt = \int_a^b \frac{t-b}{a-b}\,dt = \frac{b-a}{2}$$

$$\omega_2 = \int_a^b L_1(t)\,dt = \int_a^b \frac{t-a}{b-a}\,dt = \frac{b-a}{2}$$

$n = 3$ Simpson rule

$$\int_a^b f(t)\,dt \approx \frac{b-a}{6}\left(f(a) + 4 f\left(\frac{a+b}{2}\right) + f(b)\right)$$

Polynomial QF based on Lagrange interpolation is straight-forward

BUT: Lagrange interpolation with equidistant nodes is numerically unstable

Remedy: Use Chebychev nodes instead

⇝ yield Clenshaw-Curtis QF.

Gauss Quadrature

↳ in some sense the optimal choice for polynomial QF and a given budget n of points.

This requires a concept of quality for QFs.

⟶ Order of a QF

Definition 6.3.1 (Order of a quadrature rule). The *order* of quadrature rule $Q_n : C^0([a,b]) \to \mathbb{R}$ is defined as

$$\mathit{order}(Q_n) := \max\{m \in \mathbb{N}_0 : \quad Q_n(p) = \int_a^b p(t)\,\mathrm{d}t \quad \forall p \in \mathcal{P}_m\} + 1\,, \qquad (6.11)$$

that is, as the maximal degree +1 of polynomials for which the quadrature rule is guaranteed to be exact.

$$\int_a^b f(t)\,dt \approx \sum_{j=1}^{n} \omega_j f(c_j)$$

↑ up to which degree is this exact for polynomials

Note: This concept of order of a QF is invariant under affine transformations (such as Φ from before to transform from $[-1, 1]$ to $[a, b]$).

Example: Suppose we have a polynomial QF with n points. What can we say about its order?

Polynomial QF with n points: is based on polynomial interpolation with n nodes

So: any $p \in \mathbb{P}_{n-1}$ is uniquely determined by n values $p(t_1), \dots, p(t_n)$

$\Rightarrow$ interpolation gives p.

$\Rightarrow$ quadrature is exact for any $p \in \mathbb{P}_{n-1}$

This means $\text{order}(QF_{poly}) \geq (n-1)+1 = n$

[In fact, we will see that $> n$ is possible]

Characterization of n-point QFs with order $\geq n$:

Theorem 6.3.1 (Sufficient order conditions for quadrature rules). *An n-point quadrature rule on $[a,b]$ (see Definition 6.1.1)*

$$Q_n(f) := \sum_{j=1}^{n} w_j f(t_j)\,, \quad f \in C^0([a,b])\,,$$

with nodes $t_j \in [a,b]$ and weights $w_j \in \mathbb{R}$, $j = 1,\ldots,n$, has order $\geq n$, if and only if

$$\boxed{w_j = \int_a^b L_{j-1}(t)\,dt}\,, \quad j = 1,\ldots,n\,,$$

where L_k, $k = 0,\ldots,n-1$, is the k-th Lagrange polynomial (5.14) *associated with the ordered node set $\{t_1, t_2, \ldots, t_n\}$.*

$$L_i(t) = \prod_{\substack{j=0 \\ j\neq i}}^{n-1} \frac{t-t_j}{t_i-t_j}$$

← determined by the nodes $t_0, \ldots, t_{n-1}$

Therefore: For QF Q_n to have order $\geq n$ weights w_j only depend on node set $\mathcal{T} = \{t_1, \ldots, t_n\}$

Proof of Thm 6.3.1: Exercise

The big question: Are there n-point QFs with order $> n$?

Upper bound to the order of an n-point QF:

Theorem 6.3.2 (Maximal order of n-point quadrature rule). *The maximal order of an n-point quadrature rule is* $2n$.

Why? To see this, we construct a polynomial $q \in \mathcal{P}_{2n}$ s.t. $Q_n(q) \neq \int_a^b q(t)\,dt$.

Because then: order $< 2n+1$

or $\leq 2n$.

<u>How?</u>

$$Q_n(f) := \sum_{j=1}^{n} \omega_j^n f(c_j^n)$$

with choice

$$q(t) := (t-c_1^n)^2 (t-c_2^n)^2 \cdot \ldots \cdot (t-c_n^n)^2 \in \mathbb{P}_{2n}$$

$$= t^{2n} + \ldots$$

$q(t) \geq 0$ (by definition)

$$\Rightarrow \int_a^b q(t)\,dt > 0$$

On the other hand:

$$Q_n(q) = \sum_{j=1}^{n} \omega_j^n q(c_j^n) = 0$$

$$\Rightarrow Q_n(q) \neq \int_a^b q(t)\,dt$$

$\Rightarrow$ order$(Q_n) < 2n+1$

$\Rightarrow$ order$(Q_n) \leq 2n$. $\square$

<u>Example:</u> 2-point QF Q_2 of order 4 on $[-1,1]$

$$Q_2(p) = \int_a^b p(t)\,dt \quad \forall\ p \in \mathbb{P}_3$$

$$\Leftrightarrow \quad Q_2(\{t \mapsto t^q\}) = \frac{1}{q+1}\left(b^{q+1} - a^{q+1}\right)$$

$q = 0, 1, 2, 3$

This is sufficient because monomials $\left(\{t \mapsto t^q\}\right)_{q=0}^{n}$ form a basis of $\mathcal{P}_n$.

$a=-1,\ b=1$:

QF: $\displaystyle\int_{-1}^{1} f(t)\,dt \approx \omega_1 f(c_1) + \omega_2 f(c_2)$

$$\int_{-1}^{1} 1\,dt = 2 \overset{QF}{=} \omega_1 + \omega_2 \qquad (1)$$

$$\int_{-1}^{1} t\,dt = \left.\frac{t^2}{2}\right|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0 = \omega_1 c_1 + \omega_2 c_2 \qquad (2)$$

$$\int_{-1}^{1} t^2\,dt = \left.\frac{t^3}{3}\right|_{-1}^{1} = \frac{2}{3} = \omega_1 c_1^2 + \omega_2 c_2^2 \qquad (3)$$

$$\int_{-1}^{1} t^3\,dt = 0 = \omega_1 c_1^3 + \omega_2 c_2^3 \qquad (4)$$

Simple check: $c_1, c_2, \omega_1, \omega_2$ are all $\neq 0$

$(2) \Rightarrow \quad \omega_1 = -\frac{c_2}{c_1}\omega_2$

$(4) \Rightarrow \quad \omega_1 = -\frac{c_2^3}{c_1^3}\omega_2$

$\Rightarrow \quad \frac{c_2}{c_1} = \frac{c_2^3}{c_1^3} \quad \Rightarrow \quad \frac{c_2^2}{c_1^2} = 1$

$\Rightarrow \quad c_1^2 = c_2^2$

Use this in (3):

$$\frac{2}{3} = c_1^2 \underbrace{(\omega_1 + \omega_2)}_{\overset{(1)}{=}2}$$

$\Rightarrow \quad c_1^2 = \frac{1}{3}, \quad c_2^2 = \frac{1}{3}$

Choose c_1, c_2 s.t. $c_1 \neq c_2 \Rightarrow c_1 = -c_2$

(2) : $\omega_1 c_1 + \omega_2 c_2 = 0$

$\omega_1 - \omega_2 = 0$

$\Rightarrow \underline{\omega_1 = \omega_2}$

Pick $\underline{\underline{c_1 = -\frac{1}{\sqrt{3}}, \quad c_2 = \frac{1}{\sqrt{3}}}}$

(1) : $\omega_1 + \omega_2 = 2$

$\Rightarrow \underline{\underline{\omega_1 = 1, \quad \omega_2 = 1}}$

Q_2: 2-point QF of order 4 :

$$Q_2(f) = 1 \cdot f\left(-\frac{1}{\sqrt{3}}\right) + 1 \cdot f\left(\frac{1}{\sqrt{3}}\right)$$

More generally we can ask:

For any $n \in \mathbb{N}$, is there Q_n s.t.
Q_n is n-point and
of order $2n$?

Theorem 6.3.3 (Existence of n-point quadrature formulas of order $2n$). *Let $\{\overline{P}_n\}_{n\in\mathbb{N}_0}$ be a family of non-zero polynomials that satisfies*

- $\overline{P}_n \in \mathcal{P}_n$,
- $\int_{-1}^{1} q(t)\bar{P}_n(t)\,\mathrm{d}t = 0$ *for all* $q \in \mathcal{P}_{n-1}$ $\quad$ ($L^2([-1,1])$-orthogonality),
- *The set* $\{c_j^n\}_{j=1}^m$, $\underline{\underline{m}} \le n$, *of real zeros of* $\bar{P}_n$ *is contained in* $[-1,1]$.

Then the quadrature rule (see Definition 6.1.1) $\quad Q_n(f) := \sum_{j=1}^{m} w_j^n f(c_j^n)$

with weights chosen according to Theorem 6.3.1 provides a quadrature formula of order $2n$ on the interval $[-1,1]$.

We will see : $m = n$ $\qquad \overline{P}_n(t) = (t-c_1^n)\cdot\ldots\cdot(t-c_n^n)$

n-point QF of order $2n$

It turns out: Up to scaling factors, the $\bar{P}_n$'s are the Legendre polynomials

Definition 6.3.2 (Legendre polynomials)**.** The n-th *Legendre polynomial* P_n is defined by

- $P_n \in \mathcal{P}_n$,
- $\int_{-1}^{1} P_n(t)q(t)\,dt = 0 \quad \forall q \in \mathcal{P}_{n-1}$, $\qquad \langle P_n, q\rangle_{L^2([-1,1])} = 0$
- $P_n(1) = 1$.

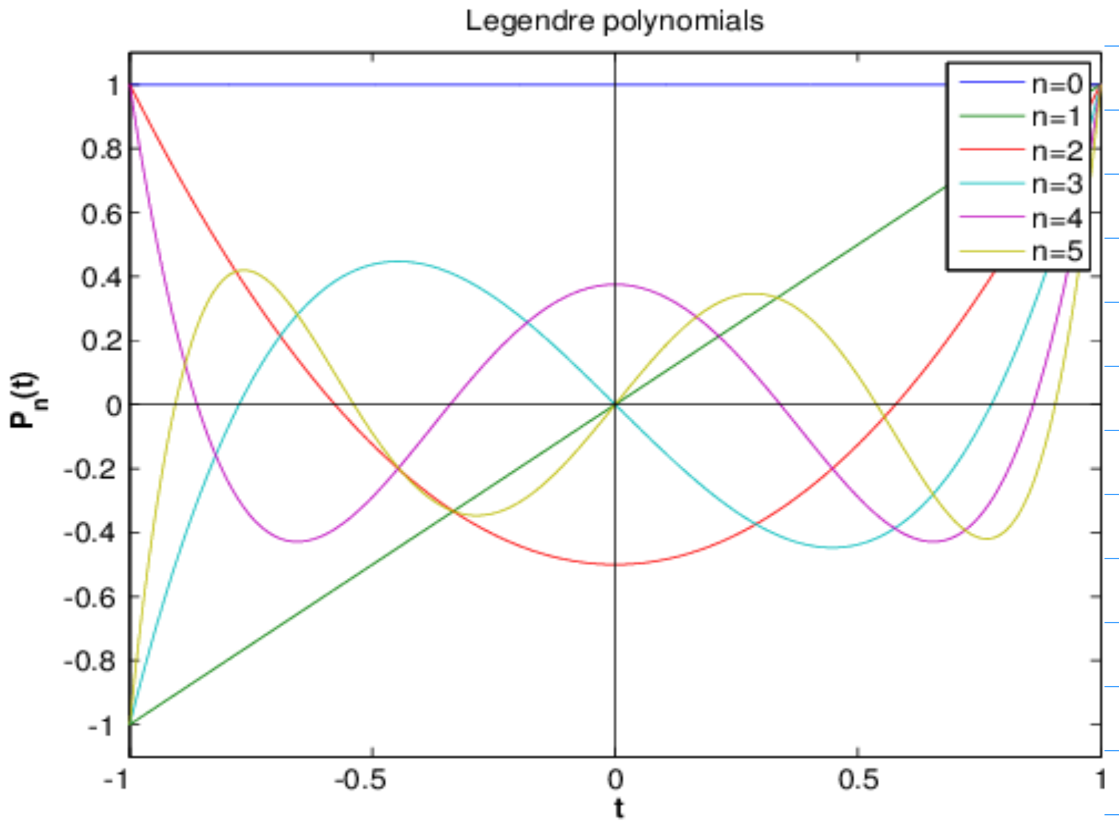


Fig. 265

Fact: P_n has n distinct zeros in $(-1,1)$

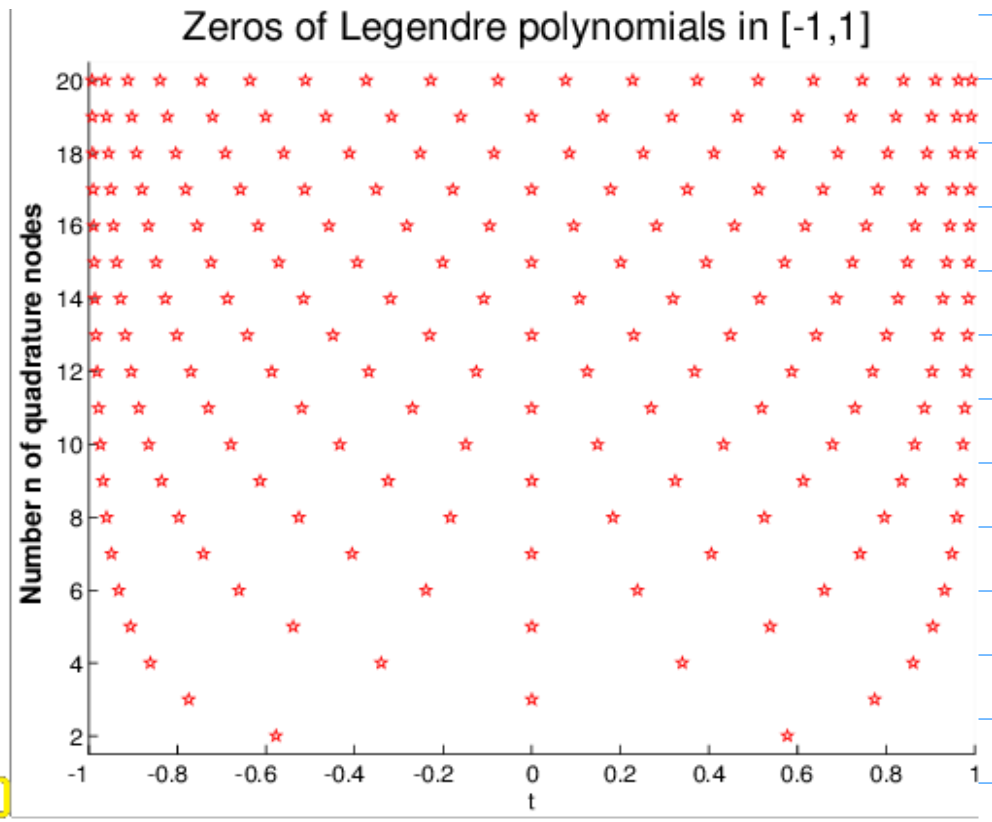


Fig. 266

Why does P_n have n distinct zeros in $(-1,1)$?

What happens if P_n has only $m<n$ zeros?

$\xi_1, \dots, \xi_m \in (-1,1)$

Observe: P_n changes sign at $\xi_1, \dots, \xi_m$.

Now define $\displaystyle q(t) := \prod_{j=1}^{m} (t - \xi_j) \in \mathcal{P}_m \subseteq \mathcal{P}_{n-1}$

and we know by definition of P_n:

$$\int_{-1}^{1} P_n(t)\, q(t)\, dt = 0 \qquad \langle P_n, q \rangle_{L^2([-1,1])}$$

BUT: q also changes sign at $\xi_1, \dots, \xi_m$!

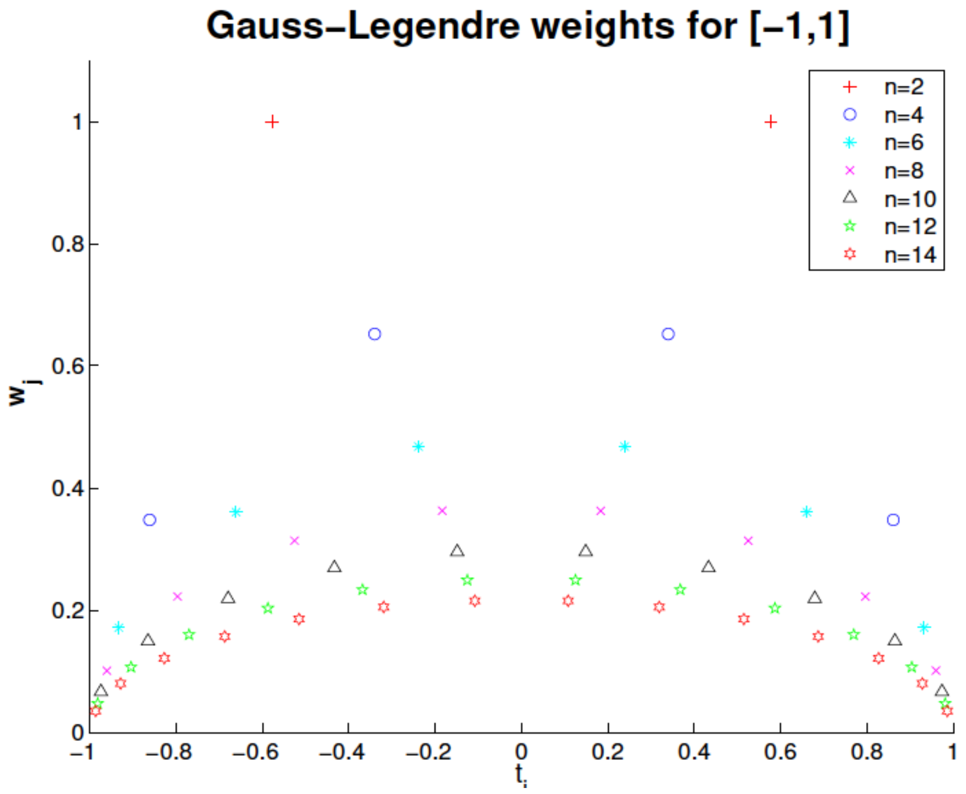
$P_n(t) \cdot q(t)$ cannot change sign on $(-1,1)$

$\Rightarrow \quad P_n \cdot q \geq 0$ on $(-1,1)$ or $P_n \cdot q \leq 0$ on $(-1,1)$

$$\Rightarrow \int_{-1}^{1} P_n(t)\, q(t)\, dt \neq 0 \qquad \lightning$$

Definition 6.3.3 (Gauss-Legendre quadrature formulas). The n-point Quadrature formulas whose nodes are given by the zeros of the n-th Legendre polynomial (see Definition 6.3.2), and whose weights are chosen according to Theorem 6.3.1, are called *Gauss-Legendre quadrature formulas*.

Note: The weights of the G-L QFs are positive.



Derivation of positivity:

We denote the Gauss points by ξ_j^n $j=1,\dots,n$ for an n-point G-L QF.

Define $q_k(t) := \prod_{\substack{j=1\\ j\neq k}}^{n} (t-\xi_j^n)^2$

$\Rightarrow$ $q_k \in \mathcal{P}_{2n-2}$ $\Rightarrow$ n-point G-L QF integrates q_k exactly!

$$0 < \int_{-1}^{1} q_k(t)\,dt = \underbrace{\sum_{j=1}^{n} \omega_j^n \cdot q_k(\xi_j^n)}_{QF} = \omega_k^n \underbrace{q_k(\xi_k^n)}_{>0}$$

$\Rightarrow$ $\omega_k^n > 0$. This is true for all $k=1,\dots,n$ $\square$.

As for Chebychev polynomials, we again have a 3-term recursion formula:

Recursive formula for Legendre polynomials

Legendre polynomials satisfy the 3-term recursion (similar to Chebychev polynomials).

$$P_{n+1}(t) := \frac{2n+1}{n+1}\, t P_n(t) - \frac{n}{n+1} P_{n-1}(t) \quad , \quad P_0 := 1\,, \quad P_1(t) := t\,. \tag{6.14}$$

Legendre polynomials: L^2-orth. polynomials

Chebychev polynomials: orth. polyn. w.r.t. weighted L^2.

Simple bound on the quadrature error:

(using positivity of weights)

Theorem 6.3.4 (Quadrature error estimate for quadrature rules with positive weights). *For every n-point quadrature rule Q_n as in (6.1) of order $q \in \mathbb{N}$ with weights $w_j \geq 0$, $j = 1, \dots, n$ the quadrature error satisfies*

$$E_n(f) := \left| \int_a^b f(t)\,dt - Q_n(f) \right| \leq 2|b-a| \underbrace{\inf_{p \in \mathcal{P}_{q-1}} \|f - p\|_{L^\infty([a,b])}}_{\text{best approximation error}} \quad \forall f \in C^0([a,b]) . \tag{6.15}$$

(Key ingredient: $\sum_{j=1}^{n} |w_j| = \sum_{j=1}^{n} w_j$)

Question: Asymptotic behavior of $E_n(f)$ as $n \to \infty$

quadrature error

Lemma 6.3.3 (Quadrature error estimates for C^r-integrands). *For every n-point quadrature rule Q_n as in (6.1) of order $q \in \mathbb{N}$ with weights $w_j \geq 0$, $j = 1, \dots, n$ we find that the quadrature error $E_n(f)$ for an integrand $f \in C^r([a,b])$, $r \in \mathbb{N}_0$, satisfies*

$$\text{in the case } q \geq r: \quad E_n(f) \leq C q^{-r} |b-a|^{r+1} \left\| f^{(r)} \right\|_{L^\infty([a,b])} , \tag{6.16}$$

$$\text{in the case } q < r: \quad E_n(f) \leq \frac{|b-a|^{q+1}}{q!} \left\| f^{(q)} \right\|_{L^\infty([a,b])} , \tag{6.17}$$

with a constant $C > 0$ independent of n, f, and $[a,b]$.

$f \in C^r([a,b])$: $E_n(f) = O(n^{-r})$ alg. conv.

If $f \in C^\infty([a,b])$: $E_n(f) = O(\lambda^n)$ exp. conv.

$\lambda \in (0,1)$

Example: 2 functions — smooth / only cont.

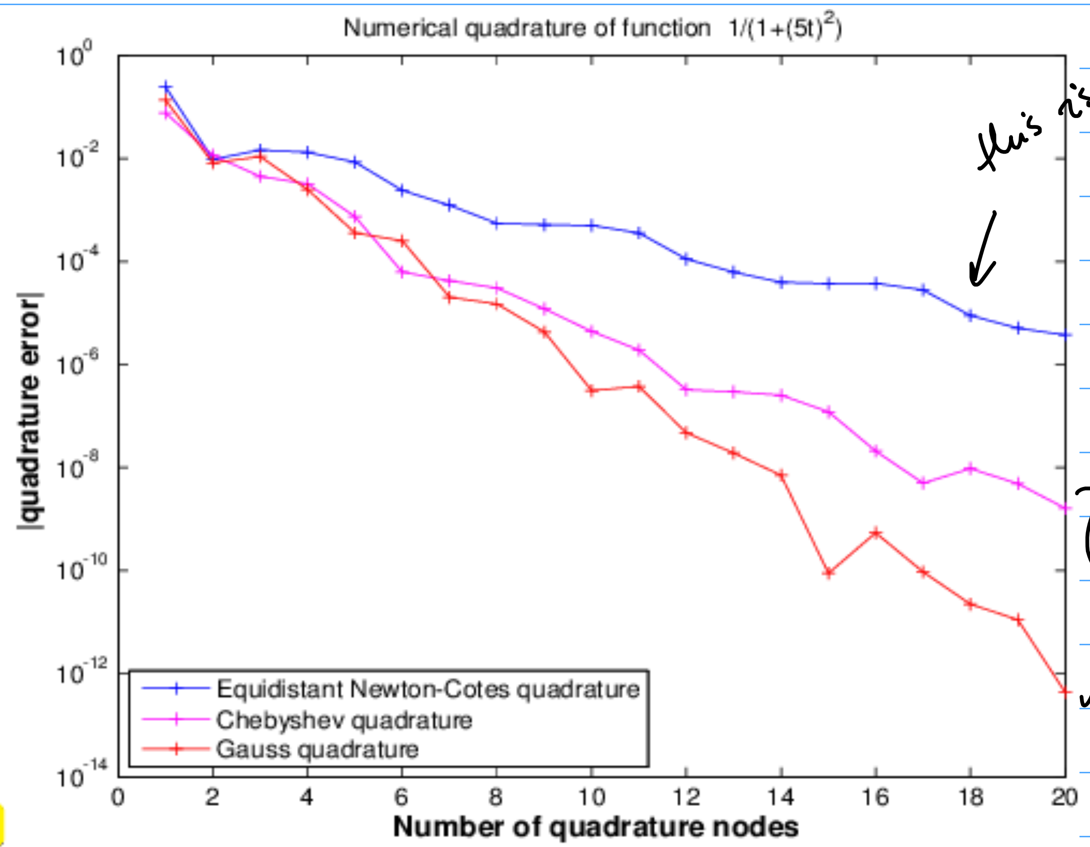


Fig. 268

quadrature error, $f_1(t) := \frac{1}{1+(5t)^2}$ on $[0,1]$

lin-log-plot

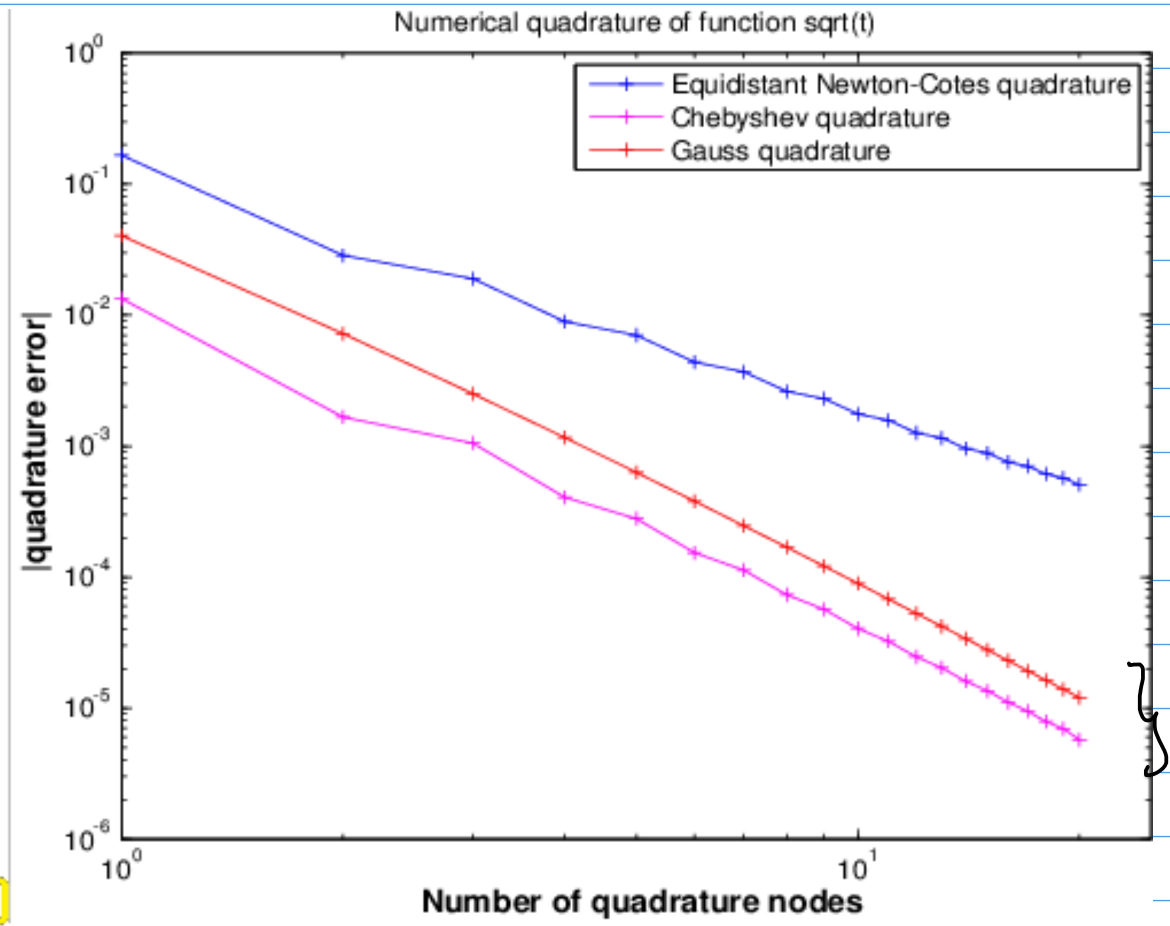


Fig. 269

quadrature error, $f_2(t) := \sqrt{t}$ on $[0,1]$

log-log-plot

algebraic conv

Remark:

The integral of this particular function can be transformed:

substitute $s = \sqrt{t}$

$dt = 2s\,ds$

$$\int_0^1 s\cdot 2s\,ds = \underbrace{\int_0^1 \overbrace{2s^2}^{\in C^\infty} ds}_{\text{apply a QF here}}$$

More generally:

$$\int_0^b \sqrt{t}\,\overbrace{g(t)}^{\in C^\infty}\,dt = \int_0^{\sqrt{b}} \underbrace{2s^2\, g(s^2)}_{\in C^\infty([0,b])}\,ds$$

What do the asymptotics actually tell us?

Suppose for fixed f we have sharp asymptotics

$$\underset{\substack{\uparrow \\ \text{alg.}}}{E_n(f)} = \mathcal{O}(n^{-r}) \implies E_n(f) \approx \underset{\substack{\uparrow \\ C>0 \text{ ind. of } n}}{C}\cdot n^{-r}$$

Now: We want to decrease the quadrature error by factor $\rho > 1$

Then: $$\frac{C\cdot n_{old}^{-r}}{C\cdot n_{new}^{-r}} = \rho \implies \boxed{n_{new} = n_{old}\cdot \rho^{1/r}}$$

This tells how many more points we need to take to improve quadrature error by a factor ρ.

Suppose we had $\overset{\text{sharp}}{\checkmark}$ exp. conv.:

$$E_n(f) = \Theta(\lambda^n) \Rightarrow E_n(f) \approx \underset{\text{ind. of } n}{\underset{\uparrow}{C}} \cdot \lambda^n$$

$$\frac{C \cdot \lambda^{n_{old}}}{C \cdot \lambda^{n_{new}}} = \rho \Rightarrow$$

$$\boxed{n_{new} = n_{old} + \left\lceil \left| \frac{\log \rho}{\log \lambda} \right| \right\rceil}$$

Here: only need to add a fixed number of nodes to improve the quadrat. error by a factor ρ.

Composite Quadrature

As for interpolation: introduce a mesh

& then apply a QF on each cell

Mesh $\mathcal{M} = \{ a = x_0 < x_1 < \ldots < x_m = b \}$

$$\int_a^b f(t)\,dt = \sum_{j=1}^{m} \int_{x_{j-1}}^{x_j} f(t)\,dt$$

on each interval $I_j := [x_{j-1}, x_j]$ apply an n_j-point QF

total number of function evaluations f: $\sum_{j=1}^{m} n_j$

Motivation: • nodes are not freely choosable (then: equidistant + global polyn. interp. is a bad idea)

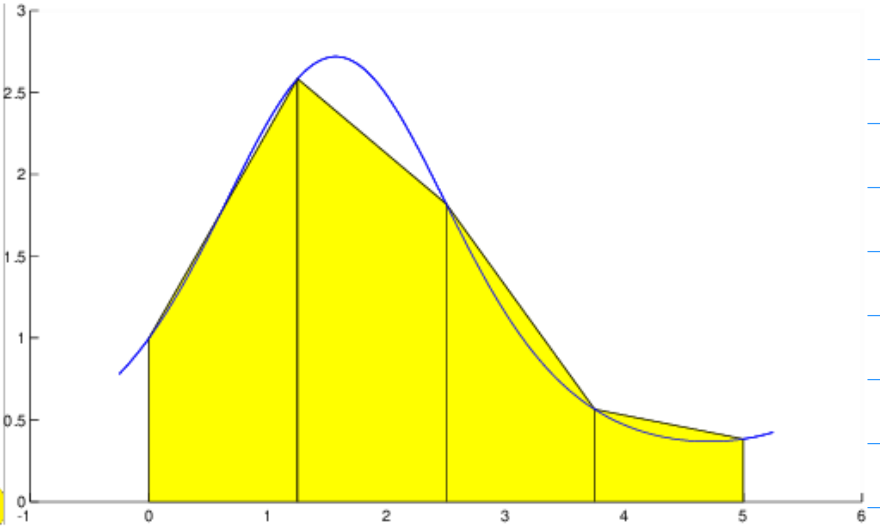
• function is e.g. highly oscillatory

Examples:

Composite trapezoidal rule, *cf.* (7.2.5)

$$\int_a^b f(t)\,\mathrm{d}t = \tfrac{1}{2}(x_1 - x_0) f(a) + \sum_{j=1}^{m-1} \tfrac{1}{2}(x_{j+1} - x_{j-1}) f(x_j) + \tfrac{1}{2}(x_m - x_{m-1}) f(b)\,. \tag{7.4.4}$$

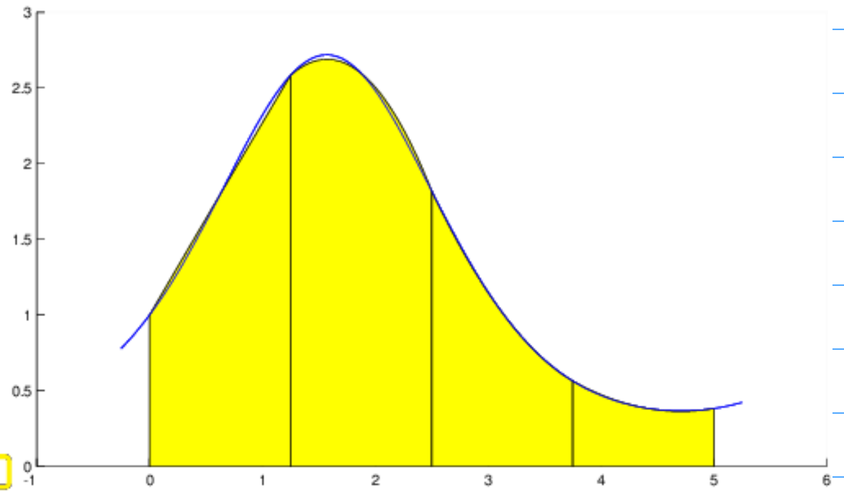
Fig. 270



Composite Simpson rule, *cf.* (7.2.6)

$$\int_a^b f(t)\,\mathrm{d}t = \tfrac{1}{6}(x_1 - x_0) f(a) + \sum_{j=1}^{m-1} \tfrac{1}{6}(x_{j+1} - x_{j-1}) f(x_j) + \sum_{j=1}^{m} \tfrac{2}{3}(x_j - x_{j-1}) f\left(\tfrac{1}{2}(x_j + x_{j-1})\right) + \tfrac{1}{6}(x_m - x_{m-1}) f(b)\,. \tag{7.4.5}$$

Fig. 271



Error estimates for composite QF:

$\rightarrow$ add errors on each I_j

Suppose on each I_j: QF $Q_{n_j}^j$ of order q & positive weights

If $f \in C^r([x_{j-1}, x_j])$:

$$\left| \int_{x_{j-1}}^{x_j} f(t)\,dt - Q_{n_j}^j (f|_{I_j}) \right| \underset{\uparrow \atop (6.17)}{\leq} C \cdot h_j^{\min\{r,q\}+1} \cdot \| f^{(\min\{r,q\})} \|_{L^\infty(I_j)}$$

$$\Rightarrow \left| \sum_{j=1}^m \left\{ \int_{x_{j-1}}^{x_j} f(t)\,dt - Q_{n_j}^j(f) \right\} \right| \leq \sum_{j=1}^m \left| \int_{x_{j-1}}^{x_j} f(t)\,dt - Q_{n_j}^j(f) \right|$$

$$\leq C \cdot \sum_{j=1}^m h_j^{\min\{r,q\}+1} \| f^{(\min\{r,q\})} \|_{L^\infty(I_j)}$$

$$\underset{\uparrow \atop h_{\mathcal{M}} = \max h_j}{\leq} C \cdot h_{\mathcal{M}}^{\min\{r,q\}} \max_{j=1,\dots,m} \| f^{(\min\{r,q\})} \|_{L^\infty(I_j)} \cdot \underbrace{\sum h_j}_{|b-a|}$$

$\Rightarrow$ algebraic convergence in mesh width $h_{\mathcal{M}}$

"h-convergence"

For $r > q$: algebraic convergence in $h_{\mathcal{M}}$ of rate q (= order of the QF)

Example: Composite trapezoidal: $q = 2$

Simpson: $q = 4$ (higher than expected)

For suff. many times differentiable functions (i.e. $r > q$): $\underset{\uparrow \atop \text{trap.}}{O(h^2)}$ vs. $\underset{\uparrow \atop \text{Simpson}}{O(h^4)}$

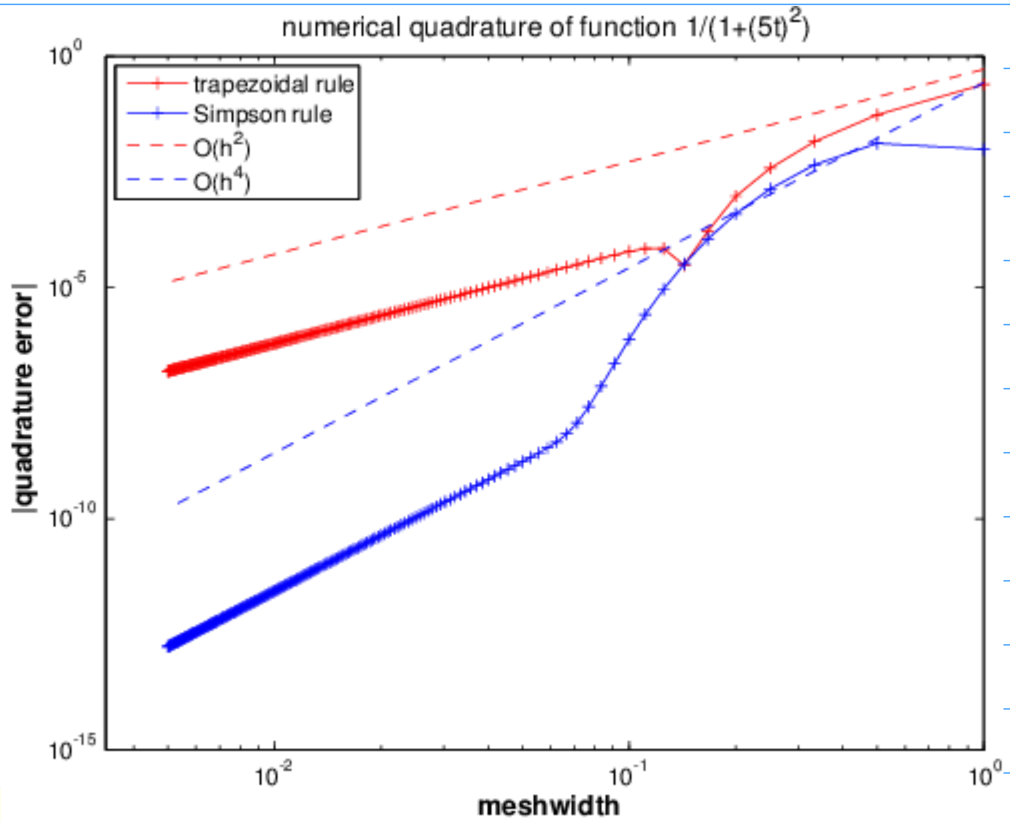


Fig. 272

log - log - plot

quadrature error, $f_1(t) := \frac{1}{1+(5t)^2}$ on $[0,1]$

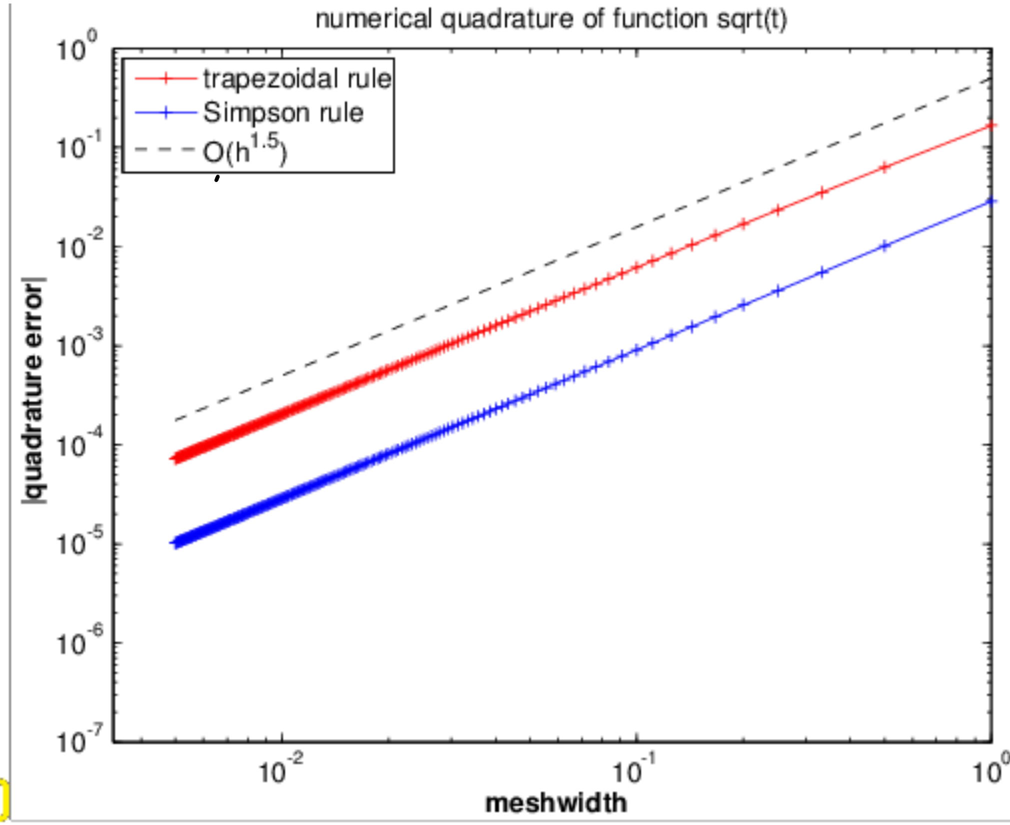


Fig. 273

$\mathcal{O}(h^{3/2})$

quadrature error, $f_2(t) := \sqrt{t}$ on $[0,1]$

$\in C^0([0,1])$

Comparison of asymptotic rates of

composite QF / global G-L QF

$f \in C^r([a,b])$: Composite QF: $\Theta(n^{-\min\{r,q\}})$

Gauss QF: $\Theta(n^{-r})$

$\Rightarrow$ Gauss is at least as good as composite QF & achieves best possible rate

$f \in C^\infty([a,b])$: composite QF: $\Theta(n^{-q})$ alg. conv.

Gauss QF: $\Theta(\lambda^n)$ exp. conv.

$\lambda \in (0,1)$